NEUMANN-BOUNDARY STABILIZATION OF THE WAVE EQUATION WITH INTERNAL DAMPING CONTROL AND APPLICATIONS

by

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عمادة الدراسات العليا
DEANSHIP OF GRADUATE STUDIES

To The Prophet ***MUHAMMAD -PBUH-***

To my home country ***Palestine***

To my dear parents

To my dear brothers

To my dear relatives

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THESIS ABSTRACT

NAME: Saed Ali Mara'Beh

TITLE OF STUDY: Neumann-Boundary Stabilization of the Wave equation with Internal Damping Control and Applications

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This thesis is devoted to the Neumann boundary stabilization of a non-homogeneous n-dimensional wave equation subject to static or dynamic boundary conditions. Using a linear feedback law involving only an internal term, we prove the well-posedness of the considered systems and provide a simple method to obtain an asymptotic convergence result for the solutions. The method consists of proposing a new energy norm, and applying the semigroup theory and LaSalle's principle. Finally, the method presented in this work is also applied to several distributed parameter systems such as the Petrovsky system, coupled wave-wave equations and elastic system.

ملخص الرسالة

الاسم الكامل: سائد علي عايش مراعبه

عنوان الرسالة: الاستقرار بحد نيومان لمعادلة الامواج مع تحكم تخامدي وتطبيقات

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هذه الاطروحة تكرس الاستقرار الحيدي لمعادلة الامواج غير المتجانسة في البعد n والتي تخضع للشروط الحدية السكونية والديناميكية باستخدام قانون التغذية الراجعة الخطي الذي ينطوي فقط على الحد التخامدي.
نقدم طريقة بسيطة لنحصل على تقارب للحلول في الأنظمة التي يتم دراستها من خلال تقديم معيار طاقة جديد.
وأخيراً، الطريقة المعروضة في هذا العمل تطبق على عدة أنظمة مثل نظام بيتروفسكي، معادلات موجة-موجة ، وأنظمة المرونة.

CHAPTER 1

INTRODUCTION

Let Ω be a bounded open connected set in $\mathbb{R}$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. Given a partition (Γ_0, Γ_1) of Γ, consider the following equation:

$$y_{tt}(x,t) - Ay(x,t) = 0, \qquad \text{in } \Omega \times (0,\infty) \tag{1.1}$$

with either static Neumann boundary conditions and initial conditions

$$\begin{cases} \partial_A y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ \partial_A y(x,t) = U(t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega), \quad y_t(x,0) = z_0(x) \in L^2(\Omega), \end{cases} \tag{1.2}$$

or dynamic Neumann boundary conditions and initial conditions

$$\begin{cases} m(x)y_{tt}(x,t) + \partial_A y(x,t) = 0, & \text{on } \Gamma_0 \times (0,\infty) \\ M(x)y_{tt}(x,t) + \partial_A y(x,t) = U(t), & \text{on } \Gamma_1 \times (0,\infty) \\ y(x,0) = y_0(x) \in H^1(\Omega), \quad y_t(x,0) = z_0(x) \in L^2(\Omega), \\ y_t|_{\Gamma_0}(x,0) = w_0^0(x) \in L^2(\Gamma_0), \ \ y_t|_{\Gamma_1}(x,0) = w_1^0(x) \in L^2(\Gamma_1) \end{cases} \tag{1.3}$$

where $A = \sum\limits_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$, $\partial_A = \sum\limits_{i,j=1}^{n} a_{ij}\nu_i\partial_j$, $\partial_k = \frac{\partial}{\partial x_k}$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω and $a_{ij} \in C^1(\overline{\Omega})$, with $a_{ij} = a_{ji}$, $\forall i,j = 1, ..., n,$, and satisfying, for $\alpha_0 > 0$, $\sum\limits_{i,j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j \geq \alpha_0 \sum\limits_{i=1}^{n} \varepsilon_i^2$, $\forall\, (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^n$. Moreover, there exist two positive constants m_0 and M_1 for which $m(x) \in L^\infty(\Gamma_0)$; $m(x) \geq m_0$, $\forall x \in \Gamma_0$, and $M(x) \in L^\infty(\Gamma_1)$; $M(x) \geq M_1$, $\forall x \in \Gamma_1$.

Furthermore, U is a feedback law depending only on a damping term, that is,

$$U(t) = -a(x)y_t(x,t), \qquad (x,t) \in \Gamma_1 \times (0,\infty), \tag{1.4}$$

where the function a satisfies

$$a \in L^\infty(\Gamma_1); \ a(x) \geq a_0 > 0, \ \forall x \in \Gamma_1. \tag{1.5}$$

Note that Γ_1 is supposed to be nonempty ($vol(\Gamma_1) \neq 0$) whereas Γ_0 may be empty.

In the paper for B. Chentouf and A. Guesmia, Neumann-Boundary Stabilization of the Wave equation and Applications, Communications in Applied Analysis 14 (2010), no 4, 541-566.(see [13]), it is proved that the solutions of each of the above systems (1.1)-(1.2) and (1.4) as well as (1.1)-(1.3) and (1.4) asymptotically tend towards a constant depending on the corresponding initial data; that is

(i) For any initial data $(y_0, z_0) \in H^1(\Omega) \times L^2(\Omega)$, the solution of the systems (1.1)-(1.2) and (1.4) satisfy: $(y(t), y_t(t)) \to (\chi, 0)$ in $H^1(\Omega) \times L^2(\Omega)$ as $t \to \infty$, where

$$\chi = \left(\int_{\Gamma_1} a \, d\sigma \right)^{-1} \left(\int_{\Omega} z_0 \, dx + \int_{\Gamma_1} a y_0 \, d\sigma \right). \tag{1.6}$$

(ii) The solution of the system (1.1)-(1.3) and (1.4) stemmed from any initial conditions $(y_0, z_0, w_0^0, w_1^0) \in H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_1)$ satisfy:

$$\big(y(t), y_t(t), y_t|_{\Gamma_0}(t), y_t|_{\Gamma_1}(t)\big) \to (\chi, 0, 0, 0)$$

where χ is defined by (1.6).

Also the nonlinear case of the boundary control is also treated.

There is a rich literature concerning the stabilization problem of the wave equation with static boundary conditions (1.1)-(1.2) and (1.4) (see [2], [5]- [8], [28], [30]-[33], [35], [37]- [40] and the references therein). In all references cited above, at least one of the following conditions is assumed to be satisfied:

- the equation (1.1) involves also the displacement term y.
- the stabilization feedback law $U(t)$ contains not only a boundary dissipation y_t but also a boundary displacement y.

- the first boundary condition (1.2) involves the displacement term y (the boundary condition (1.2) is replaced, for instance, by $y = 0$ or $\partial_A y + y = 0$ on $\Gamma_0 \times (0, \infty)$).

In this thesis all the results stated in [13] remain valid if the damping control ay_t is distributed, i.e., the equation (1.1) is replaced by

$$y_{tt}(x,t) \, - \, Ay(x,t) \, + \, a(x)\, y_t(x,t) \, = 0, \quad \text{in } \Omega \times (0, \infty).$$

In fact, in this case, the Neumann boundary conditions (static or dynamical) are homogeneous and one just needs to change, in the energy norm, the integral term $\int_{\Gamma_1} a(x)y(x,t)\, d\sigma$ to $\int_{\Omega} a(x)y(x,t)\, dx$ and do the appropriate modifications.

CHAPTER 2

DEFINITIONS AND ELEMENTARY PROPERTIES

2.1 L^p spaces

Definition 2.1 *[4] Let* $\Omega \subset \mathbb{R}^n$ *and* $p \in \mathbb{R}$ *with* $n \in \mathbb{N}^* = \{1, 2, 3, ...\}$ *and* $1 \leq p < \infty$*; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R};\ f \text{ is measurable and } \int\limits_{\Omega} |f(x)|^p d\mu < \infty \right\},$$

endowed with the norm

$$\|f\|_{L^p} = \|f\|_p = \left[\int\limits_{\Omega} |f(x)|^p d\mu \right]^{1/p}.$$

Definition 2.2 *[4] We set*

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \\ \text{such that } |f(x)| \leq C, \text{ a.e., on } \Omega \end{array} \right\},$$

endowed with the norm

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf\{C;\ |f(x)| \le C \text{ a.e., on } \Omega\}.$$

Remark: If $f \in L^\infty(\Omega)$, then we have $|f(x)| \le \|f\|_\infty$ a.e., on Ω.

Notation: Let $1 \le p \le \infty$, we denote by p' the conjugate exponent defined by $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 2.1 *[4]* (Hölder′s inequality). *Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \le p \le \infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \le \|f\|_p \|g\|_{p'}.$$

2.2 Hilbert spaces

Definition 2.3 *[25] A vector space (or linear space) consists of the following:*

1. A field F of scalars.

2. A set V of vectors.

3. An operation called vector addition, which associates with each of vectors u, v in V a vector $u + v$ called the sum of u and v, in such a way that

(a) Addition is commutative: $u + v = v + u$.

(b) Addition is associative: $u + (v + w) = (u + v) + w$.

(c) There is a unique vector 0 in V, called the zero vector, such that $u + 0 = u$ for all u in V.

(d) For each vector u in V, there is a unique vector $-u$ in V such that $u + (-u) = 0$.

4. An operation called scalar multiplication, which associates with each scalar c in F and a vector u in V a vector $c\,u$ in V, called the product of c and u, in such a way that

(a) $1u = u$ *for every* u *in* V.

(b) $(c_1c_2)u = c_1(c_2u)$.

(c) $c(u+v) = cu + cv$.

(d) $(c_1 + c_2)u = c_1u + c_2u$.

Definition 2.4 *[25]* ***A bilinear form*** *on a vector space* V *(over the field* F*) is a function* $f : V \times V \to F$ *such that* $f(cu+v, w) = cf(u, w) + f(v, w)$ *and* $f(u, cv + w) = cf(u, v) + f(u, w)$ *for all* u, v, w *in* V *and* c *in* F. *The set of all bilinear forms on* V *is a vector space over* F *denoted by* $L(V, V, F)$ *or* $B(V)$.

Definition 2.5 *[4] Let* H *be a vector space. A scalar product* (u, v) *is a bilinear form on* $\mathrm{H} \times \mathrm{H}$ *with values in* $\Re$ *(i.e., a map from* $\mathrm{H} \times \mathrm{H}$ *to* $\Re$ *that is linear in both variables) such that*

$$(u, v) = (v, u) \qquad \forall u, v \in \mathrm{H} \quad \textit{(symmetry)}$$

$$(u, u) \geq 0 \qquad \forall u \in \mathrm{H} \quad \textit{(positive)}$$

$$(u, u) \neq 0 \qquad \forall u \neq 0 \quad \textit{(definite)}$$

Let us recall that a scalar product satisfies the Cauchy-Schwarz inequality

$$|(u, v)| \leq (u, u)^{1/2}(v, v)^{1/2} \qquad \forall u, v \in \mathrm{H}.$$

It follows from Cauchy-Schwarz inequality that the quantity

$$\|u\| = (u,u)^{1/2}$$

is a norm we shall often denote it by $\| \ \|$ or $| \ |$.

Indeed, we have

$$\|u+v\|^2 = (u+v, u+v) = \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \leq \|u\|^2 + 2\,\|u\|\,\|v\| + \|v\|^2,$$

and thus $\|u+v\| \leq \|u\| + \|v\|$.

Let us recall the classical parallelogram law:

$$\left\|\tfrac{a+b}{2}\right\|^2 + \left\|\tfrac{a-b}{2}\right\|^2 = \tfrac{1}{2}(\|a\|^2 + \|b\|^2) \qquad \forall a,b \in \mathrm{H}.$$

Definition 2.6 *[4] A Hilbert space is a vector space* H *equipped with a scalar product such that* H *is complete for the norm* $\| \ \|$.

(Through this chapter, H denotes a Hilbert space).

Basic example: $L^2(\Omega)$ equipped with the scalar product

$$(u,v) = \int_{\Omega} u(x)v(x)\,d\mu$$

is a Hilbert space.

Definition 2.7 *[4] A bilinear form* $a : \mathrm{H} \times \mathrm{H} \to \Re$ *is said to be*

(i) Continuous if there is a constant C *such that*

$$|a(u,v)| \leq C\,\|u\|\,\|v\| \qquad \forall u,v \in \mathrm{H}.$$

(ii) Coercive if there is a constant $\alpha > 0$ *such that*

$$a(v,v) \geq \alpha \|v\|^2 \qquad \forall v \in \mathrm{H}.$$

Theorem 2.2 *[4] **(Lax-Milgram).** Assume that a is a continuous coercive bilinear form on $H \times H$. Then, given any $\varphi \in \mathrm{H}^*$, there is a unique element $u \in \mathrm{H}$ such that $a(u, v) = \langle \varphi, v \rangle$, $\forall v \in \mathrm{H}$, where H^* is the dual space of H.*

2.3 The Hille-Yosida theorem

Definition 2.8 *[4] A (bounded or unbounded) linear operator $A : D(A) \subset \mathrm{H} \to \mathrm{H}$ is said to be **monotone** if it satisfies*

$$(Av, v) \geq 0 \qquad \forall v \in D(A).$$

*It is called **maximal monotone** if it is monotone and $R(I+A) = \mathrm{H}$, i.e., $\forall f \in \mathrm{H}$* $\exists u \in D(A)$ such that $u + Au = f$.

Theorem 2.3 *[4] **(Hille-Yosida).** Let A be a maximal monotone operator. Then,*

1) Given any $u_0 \in D(A)$, there exists a unique function $u \in C^1([0,\infty), \mathrm{H}) \cap C([0,\infty), D(A))$ satisfying

$$\begin{cases} \frac{du}{dt} + Au = 0 \qquad on\ [0,\infty), \\ u(0) = u_0 \ . \end{cases} \tag{2.1}$$

*in the **classical** sense. Moreover,*

$$|u(t)| \leq |u_0| \ \ and \ \ \left|\tfrac{du}{dt}(t)\right| = |Au(t)| \leq |Au_0| \qquad \forall t \geq 0.$$

*2) Given any $u_0 \in \overline{D(A)}$, there exists a unique function $u \in C([0,\infty), \mathrm{H})$ satisfying (2.1) in the **weak** sense and $|u(t)| \leq |u_0|$, $\forall t \geq 0$.*

CHAPTER 3

THE WAVE EQUATION WITH STATIC BOUNDARY CONDITIONS

3.1 Introduction

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2 . Consider the following wave equation with static boundary conditions:

$$\begin{cases} y_{tt}(x,t) - Ay(x,t) + a(x)\, y_t(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ \partial_A y(x,t) = 0, & \text{on } \Gamma \times [0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x) & \text{in } \Omega\,, \end{cases} \tag{3.1}$$

where (y_0, z_0) is given initial data in $H^1(\Omega) \times L^2(\Omega)$, $A = \sum\limits_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$, $\partial_A = \sum\limits_{i,j=1}^{n} a_{ij}\nu_i\partial_j$, $\partial_k = \frac{\partial}{\partial x_k}$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω and $a_{ij} \in C^1(\overline{\Omega})$, with $a_{ij} = a_{ji}$, $\forall i, j = 1, ..., n,$, and satisfying, for $\alpha_0 > 0$, $\sum\limits_{i,j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j \geq \alpha_0 \sum\limits_{i=1}^{n} \varepsilon_i^2$, $\forall (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^n$.
Moreover, $a(x) \in L^\infty(\Omega)$ such that there exists $a_0 > 0$ for which $a(x) \geq a_0$, a.e. $x \in \Omega$.

3.2 Preliminaries and well-posedness of the problem

In this subsection, we study the existence and uniqueness of the solutions of the system (3.1). Let us consider the state space

$$\Upsilon = H^1(\Omega) \times L^2(\Omega),$$

equipped with the inner product

$$\langle (y, z), (\tilde{y}, \tilde{z}) \rangle_\Upsilon = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \, \partial_j \tilde{y} + z\tilde{z} \right) dx + \varepsilon \left(\int_\Omega (z + ay) \, dx \right) \left(\int_\Omega (\tilde{z} + a\tilde{y}) \, dx \right), \tag{3.2}$$

where ε >0 is a constant to be determined. This inner product is inspired from the approach of [13] introduced for the boundary feedback case. The first result is stated in the following proposition.

Proposition 3.1 *The state space $\Upsilon = H^1(\Omega) \times L^2(\Omega)$, endowed with the inner product (3.2) is a Hilbert space provided that ε is small enough.*

Proof. It is sufficient to show that the norm $\|.\|_\Upsilon$ induced by the inner product (3.2) is equivalent to the usual one $\| \,.\, \|_{H^1(\Omega)\times L^2(\Omega)}$; that is, we prove the existence of two positive constants K and $\tilde{K}$ such that

$$K\,\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \,\le\, \|(y,z)\|^2_\Upsilon \,\le \tilde{K}\;\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)}\,. \tag{3.3}$$

On one hand,

$$\|(y,z)\|^2_\Upsilon = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j y\right) dx \,+ \int_\Omega z^2 dx \;+\, \varepsilon \left(\int_\Omega (z+ay)dx\right)^2.$$

Applying Hölder′s inequality and Young's inequalities, we get

$$\|(y,z)\|^2_{H^1(\Omega)\times L^2(\Omega)} \le \tfrac{1}{2}\int_\Omega \sum_{i,j=1}^n \sup_{x\in\Omega}|a_{ij}(x)|\left((\partial_i y)^2 + (\partial_j y)^2\right) dx + \int_\Omega z^2 dx$$
$$+\,\varepsilon\, vol(\Omega)\int_\Omega (|z|+|a|\,|y|)^2 dx.$$

Let $a_1 = \max_{i,j}\, \sup_{x\in\Omega}|a_{ij}(x)|$ and using the fact that $2|a|\,|y|\,|z| \,\le\, \|a\|_\infty(y^2+z^2)$, we get

$$\begin{aligned}\|(y,z)\|^2_\Upsilon \le\; & \tfrac{1}{2}a_1\left(n\int_\Omega |\nabla y|^2\,dx + \,n\int_\Omega |\nabla y|^2\,dx\right) \,+\, \int_\Omega z^2 dx\\
&+\varepsilon\, vol(\Omega)\int_\Omega (z^2 + \|a\|^2_\infty\, y^2 + \|a\|_\infty y^2 + \|a\|_\infty z^2)\,dx\\
&=\; na_1\int_\Omega |\nabla y|^2 dx \,+\, \varepsilon\, vol(\Omega)\|a\|_\infty(\|a\|_\infty+1)\int_\Omega y^2 dx\\
&+\,(1+\varepsilon\, vol(\Omega)(\|a\|_\infty+1))\int_\Omega z^2\,dx.\end{aligned}$$

Let $\delta_0 = \varepsilon\, vol(\Omega)\|a\|_\infty(\|a\|_\infty+1)$, $\beta_0 = 1+\varepsilon\, vol(\Omega)(\|a\|_\infty+1)$ and $\tilde{K} = \max\{na_1,\, \delta_0,\, \beta_0\}$. Then

$$\|(y,z)\|_{\Upsilon}^2 \leq \tilde{K} \; \|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)}^2 \, . \tag{3.4}$$

On the the Other hand, using the coercivity of (a_{ij})

$$\|(y,z)\|_{\Upsilon}^2 \geq \alpha_0 \sum_{i=1}^{n} \int_{\Omega} (\partial_i y)^2 \, dx + \int_{\Omega} z^2 dx + \varepsilon \left[\left(\int_{\Omega} z \, dx \right)^2 + \left(\int_{\Omega} ay \, dx \right)^2 + 2 \left(\int_{\Omega} z \, dx \right) \left(\int_{\Omega} ay \, dx \right) \right].$$

But $2\varepsilon \left(\int_{\Omega} z \, dx \right) \left(\int_{\Omega} ay \, dx \right) \geq -\varepsilon \left[\delta \left(\int_{\Omega} ay \, dx \right)^2 + \frac{1}{\delta} \left(\int_{\Omega} z \, dx \right)^2 \right], \quad \forall \delta > 0.$

Then

$\|(y,z)\|_{\Upsilon}^2 \geq \alpha_0 \int_{\Omega} |\nabla y|^2 dx + \int_{\Omega} z^2 dx + \varepsilon(1-\delta) \left(\int_{\Omega} ay \, dx \right)^2 + \varepsilon(1 - \frac{1}{\delta}) \left(\int_{\Omega} z \, dx \right)^2.$

Using generalized Poincaré's inequality [3], we can prove that there exists a positive constant c_0 such that

$$\int_{\Omega} y^2 \, dx \leq c_0 \left[\int_{\Omega} |\nabla y|^2 \, dx + \left(\int_{\Omega} ay \, dx \right)^2 \right], \qquad \forall y \in H^1(\Omega) \tag{3.5}$$

which implies that

$$\left(\int_{\Omega} ay \, dx \right)^2 \geq \frac{1}{c_0} \int_{\Omega} y^2 dx - \int_{\Omega} |\nabla y|^2 \, dx. \tag{3.6}$$

Now, for $0 < \delta < 1$ (so $1 - \frac{1}{\delta} < 0$ and $1 - \delta > 0$)

$$\|(y,z)\|_{\Upsilon}^2 \geq (\alpha_0 - \varepsilon(1-\delta)) \int_{\Omega} |\nabla y|^2 dx + \frac{\varepsilon(1-\delta)}{c_0} \int_{\Omega} y^2 \, dx$$
$$+ \left(1 + \varepsilon \left(1 - \frac{1}{\delta}\right) vol(\Omega)\right) \int_{\Omega} z^2 dx.$$

We choose $\varepsilon > 0$ and $0 < \delta < 1$ such that the coefficients of $\int_{\Omega} |\nabla y|^2 dx$, $\int_{\Omega} y^2 dx$

and $\int\limits_{\Omega} z^2 dx$ are positive; that is,

$\alpha_0 - \varepsilon(1-\delta) > 0$, which implies that $\varepsilon < \frac{\alpha_0}{1-\delta}$.

$1 + \varepsilon(1 - \frac{1}{\delta})vol(\Omega) > 0$, then $\varepsilon < \frac{1}{\left(\frac{1}{\delta}-1\right)vol(\Omega)}$.

Because $0 < \delta < 1$ and $\alpha_0 > 0$, it is sufficient to choose $\varepsilon > 0$ such that

$0 < \varepsilon < \min\left\{\frac{\alpha_0}{1-\delta}, \frac{1}{\left(\frac{1}{\delta}-1\right)vol(\Omega)}\right\}$.

On the other hand, $c_0 > 0$, so $\frac{\varepsilon(1-\delta)}{c_0} > 0$.

Finally,

$$\|(y,z)\|_{\Upsilon}^2 \geq K\, \|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)}^2\,, \tag{3.7}$$

where $K = \min\left\{\alpha_0 - \varepsilon(1-\delta),\ \frac{\varepsilon(1-\delta)}{c_0},\ 1 + \varepsilon\left(1 - \frac{1}{\delta}\right) vol(\Omega)\right\}$.

From (3.4) and (3.7) we get

$$K\, \|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)}^2 \leq \|(y,z)\|_{\Upsilon}^2 \leq \tilde{K}\, \|(y,z)\|_{H^1(\Omega)\times L^2(\Omega)}^2\,.$$

Therefore, the state space $\Upsilon = H^1(\Omega)\times L^2(\Omega)$ endowed with the inner product (3.2) is a Hilbert space. ▌

We turn now to the formulation of the system (3.1) in an abstract form in Υ.

Let $z(t) = y_t(t)$ and $\Phi(t) = (y(t), z(t))$.

Then, the system (3.1) can be written as

$$\begin{cases} \Phi_t(t) + \mathrm{T}\Phi(t) = 0, \\ \Phi(0) = \Phi_0 = (y(0), z(0)) = (y_0, z_0), \end{cases} \tag{3.8}$$

where T is an unbounded linear operator defined by:

$$\mathrm{T}(y,z) = (-z, -Ay + az), \qquad \forall (y,z) \in D(\mathrm{T}) \tag{3.9}$$

and

$$
\begin{aligned}
D(\mathrm{T}) &= \{(y,z) \in H^1(\Omega) \times L^2(\Omega) : \; \mathrm{T}(y,z) \in H^1(\Omega) \times L^2(\Omega) \text{ and } \partial_A y = 0 \text{ on } \Gamma\} \\
&= \left\{ \begin{array}{l} (y,z) \in H^1(\Omega) \times L^2(\Omega) \; : \; (-z, -Ay + az) \in H^1(\Omega) \times L^2(\Omega) \\ \text{and } \partial_A y = 0 \text{ on } \Gamma \end{array} \right\} \\
&= \left\{ \begin{array}{l} (y,z) \in H^1(\Omega) \times L^2(\Omega) \; : \; -z \in H^1(\Omega), \; -Ay + az \in L^2(\Omega) \\ \text{and } \partial_A y = 0 \text{ on } \Gamma \end{array} \right\} \\
&= \{(y,z) \in H^2(\Omega) \times H^1(\Omega), \; \partial_A y = 0 \text{ on } \Gamma\}\,.
\end{aligned}
\tag{3.10}
$$

We prove that T is maximal monotone operator.

We have, for any $(y,z) \in D(\mathrm{T})$,

$$
\begin{aligned}
\langle \mathrm{T}(y,z), (y,z) \rangle_{\Upsilon} &= \langle (-z, -Ay + az), (y,z) \rangle_{\Upsilon} \\
&= \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij} \partial_i (-z) \partial_j y \right) dx + \int_{\Omega} (-Ay + az) z \, dx \\
&\quad + \varepsilon \left(\int_{\Omega} (-Ay + az - az) dx \right) \left(\int_{\Omega} (z + ay) dx \right).
\end{aligned}
$$

By applying the Green's formula and the fact that $\partial_A y = 0$ on Γ, we find

$\int_{\Omega} (-Ay + az - az)\, dx \;=\; - \int_{\Omega} Ay \, dx = - \int_{\Gamma} \partial_A y \; d\sigma = 0,$ and

$$
\int_{\Omega} -Ayz \, dx = \; - \int_{\Gamma} \partial_A y \, z \, d\sigma + \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij} \partial_j y \, \partial_i z \right) dx \;=\; \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij} \partial_j y \partial_i z \right) dx.
$$

Then we get $\langle \mathrm{T}(y,z), (y,z) \rangle_{\Upsilon} \;=\; \int_{\Omega} a \, z^2 \, dx \geq 0$, so we conclude that T is monotone.

Now, we prove that $Id + \mathrm{T}$ is surjective.

Let $(f_1, f_2) \in \Upsilon$. We want to find $(y,z) \in D(T)$ such that $(Id + \mathrm{T})(y,z) = (f_1, f_2)$.

We have $(Id + \mathrm{T})(y,z) = (f_1, f_2)$; that is $(y,z) + \mathrm{T}(y,z) = (f_1, f_2)$.

This means that

$$(y, z) + (-z, -Ay + az) = (f_1, f_2) \tag{3.11}$$

The first equation of (3.11) implies that $y - z = f_1$, so $z = y - f_1$.

The second equation of (3.11) becomes

$$-Ay + (a+1)z = f_2 \;\;\Leftrightarrow\;\; -Ay + (a+1)(y - f_1) = f_2,$$

which is equivalent to

$$-Ay + (a+1)y = f \;\; where \;\; f = f_2 + (a+1)f_1 \;\in L^2(\Omega). \tag{3.12}$$

Then it is sufficient to prove that (3.12) has a solution $y \in H^2(\Omega)$, satisfying

$\partial_A y = 0$ on Γ, therefore $z = y - f_1 \in H^1(\Omega)$ and (3.11) holds.

By the variational formulation [4], let y be a solution of (3.12), then $\forall \varphi \in H^1(\Omega)$,

we find

$\int\limits_\Omega (a+1)y\varphi\, dx - \int\limits_\Omega Ay\,\varphi\, dx = \int\limits_\Omega f\varphi\, dx.$

Using the Green's formula and the fact that $\partial_A y = 0 \,\text{on}\, \Gamma$, we get

$\int\limits_\Omega (a+1)\; y\varphi\, dx + \int\limits_\Omega \left(\sum\limits_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi \right) dx \;= \int\limits_\Omega f\varphi\, dx\,.$

Now, let us consider the application

$$F:\; H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$
$$(y, \varphi) \qquad \to \; F(y,\varphi) = \int\limits_\Omega \left((a+1)y\varphi + \sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi \right) dx.$$

It is clear that F is bilinear. We want to prove that F is continuous and coercive.

We have

$$|F(y,\varphi)| = \left|\int_\Omega \left((a+1)y\varphi + \sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi\right) dx\right|$$
$$\leq \int_\Omega \left((a+1)\,|y|\,|\varphi| + \left|\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi\right|\right) dx.$$

Applying Hölder′s inequality, and putting $a_1 = \max\limits_{i,j} \sup\limits_{x\in\Omega} |a_{ij}(x)|$, we find

$$|F(y,\varphi)| \leq (\|a\|_\infty + 1)\left(\int_\Omega |y|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\varphi|^2 dx\right)^{\frac{1}{2}} + na_1\left(\sum_{i=1}^n \int_\Omega |\partial_i y|^2 dx\right)^{\frac{1}{2}} \left(\sum_{j=1}^n \int_\Omega |\partial_j \varphi|^2 dx\right)^{\frac{1}{2}}$$
$$\leq \left[(\|a\|_\infty + 1)\left(\int_\Omega |y|^2 dx\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla y|^2 dx\right)^{\frac{1}{2}}\right] \times \left[\left(\int_\Omega |\varphi|^2 dx\right)^{\frac{1}{2}} + na_1\left(\int_\Omega |\nabla \varphi|^2 dx\right)^{\frac{1}{2}}\right]$$

$$\leq \left[\sqrt{2}(\|a\|_\infty + 1)\left(\int_\Omega \left(|y|^2 + |\nabla y|^2\right) dx\right)^{\frac{1}{2}}\right] \times \left[\sqrt{2}na_1\left(\int_\Omega \left(|\varphi|^2 + |\nabla \varphi|^2\right) dx\right)^{\frac{1}{2}}\right]$$
$$= 2n(\|a\|_\infty + 1)a_1 \|y\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}.$$

That is

$$|F(y,\varphi)| \leq c_1 \|y\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}, \qquad \text{where } c_1 = 2n(\|a\|_\infty + 1)a_1.$$

This means that F is continuous.

On the other hand, we have

$$F(y,y) = \int_\Omega \left((a+1)|y|^2 + \sum_{i,j=1}^n a_{ij}\partial_i y \partial_j y\right) dx \;\geq\; \int_\Omega \left((a_0+1)|y|^2 + \alpha_0 \sum_{i=1}^n (\partial_i y)^2\right) dx$$
$$\geq \min\{a_0+1, \alpha_0\} \int_\Omega \left(|y|^2 + |\nabla y|^2\right) dx$$
$$= c_2\, \|y\|^2_{H^1(\Omega)}, \quad \text{where } c_2 = \min\{a_0+1, \alpha_0\}.$$

This means F is coercive.

Now, let us consider the application

$$L:\; H^1(\Omega) \;\to\; \mathbb{R}$$
$$\varphi \quad \to\; L(\varphi) = \int_\Omega f\varphi\, dx.$$

It is clear that L is linear. On the other hand, using Hölder′s inequality, we have

$$
\begin{aligned}
|L(\varphi)| = \left|\int_\Omega f\varphi\, dx\right| &\le \int_\Omega |f|\,|\varphi|\, dx \le \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\varphi|^2 dx\right)^{\frac{1}{2}} \\
&\le \|f\|_{L^2(\Omega)} \left(\int_\Omega (|\varphi|^2 + |\nabla\varphi|^2) dx\right)^{\frac{1}{2}} \\
&= \|f\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)},
\end{aligned}
$$

which implies that

$|L(\varphi)| \le c_3 \|\varphi\|_{H^1(\Omega)}$, where $c_3 = \|f\|_{L^2(\Omega)}$. This means that L is continuous.

Applying Lax-Milgram theorem [4], we deduce that there exists a unique

$y \in H^1(\Omega)$ such that $F(y,\varphi) = L(\varphi)$, $\forall \varphi \in H^1(\Omega)$. That is

$$\int_\Omega (a+1)\, y\varphi\, dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi\right) dx = \int_\Omega f\varphi\, dx, \quad \forall \varphi \in H^1(\Omega),$$

therefore,

$$\int_\Omega (a+1)\, y\varphi\, dx + \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi\right) dx - \int_\Omega f\varphi\, dx = 0\,, \qquad \forall \varphi \in H^1(\Omega). \tag{3.13}$$

By using Green's formula and taking $\varphi \in H_0^1(\Omega)$, we find

$$\int_\Omega (a+1) y\varphi\, dx - \int_\Omega Ay\, \varphi\, dx - \int_\Omega f\varphi\, dx = 0, \ \ \forall \varphi \in H_0^1(\Omega).$$

Thus $\int_\Omega ((a+1)y - Ay - f)\, \varphi\, dx = 0$, $\forall \varphi \in H_0^1(\Omega)$;

that is,

$$\langle (a+1)y - Ay - f, \varphi\rangle_{L^2(\Omega)} = 0, \quad \forall \varphi \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, we get $(a+1)y - Ay - f = 0$ in $L^2(\Omega)$,

which is equivalent to $(a+1)y - Ay = f$ in $L^2(\Omega)$, so (3.12) holds.

Because $y \in H^1(\Omega)$ and $f \in L^2(\Omega)$, then from the elliptic regularity, we have $y \in H^2(\Omega)$. By going back to (3.13) and using Green's formula, we get

$\int\limits_{\Omega} ((a+1)y - Ay - f)\,\varphi\,dx + \int\limits_{\Gamma} \partial_A y\,\varphi\,d\sigma = 0, \qquad \forall \varphi \in H^1(\Omega).$

Then (3.12) implies that $\int\limits_{\Gamma} \partial_A y\,\varphi\,d\sigma = 0, \quad \forall \varphi \in H^1(\Omega)$. Then $\partial_A y = 0$ on Γ.

We conclude that (3.11) has a unique solution $(y, z) \in D(\mathrm{T})$, that is $Id + \mathrm{T}$ is surjective. Finally, we conclude that T is maximal monotone operator.

By Hille-Yosida theorem (see [34] and [36]), we get the following:

1) For all $\Phi_0 \in \Upsilon = H^1(\Omega) \cap L^2(\Omega)$, there exists a unique $\Phi \in C(\mathbb{R}^+, \Upsilon)$ solution of (3.8). This implies that there exists a unique solution y of (3.1) satisfying

$y \in C(\mathbb{R}^+, H^1(\Omega))\,,\ y_t \in C(\mathbb{R}^+, L^2(\Omega)) \quad \Leftrightarrow\ y \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^1(\Omega)).$

2) If $\Phi_0 \in D(\mathrm{T})$, then $\Phi \in C^1(\mathbb{R}^+, \Upsilon) \cap C(\mathbb{R}^+, D(\mathrm{T}))$; that is, for $(y_0, z_0) \in D(\mathrm{T})$, $(y, z) \in C^1(\mathbb{R}^+, H^1(\Omega) \times L^2(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega) \times H^1(\Omega))$ solution of (3.1) satisfying

$y \in C^1(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))\,,\ y_t \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^1(\Omega))$

$\Leftrightarrow\ y \in C^2(\mathbb{R}^+, L^2(\Omega)) \cap C^1(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))\,.$

3.3 Stabilization of the problem

In this subsection, we prove a stability result which is similar to the one obtained in [13] for the boundary feedback case.

Definition 3.1 *The ω-limit set is*

$$\omega(y_0,z_0)=\left\{\begin{array}{l}(\omega_1,\omega_2)\in\Upsilon\ :\ \exists\,\{t_n\}\text{ an increasing sequence of positive numbers}\,;\\ \\ \lim\limits_{n\to\infty}\|(y(t_n),z(t_n))-(\omega_1,\omega_2)\|_\Upsilon=0\end{array}\right\}.$$

Theorem 3.1 *For any initial data* $\Phi_0=(y_0,z_0)\in\Upsilon$, *the solution* $\Phi(t)=(y(t),z(t))\to(\chi,0)$ *in* Υ *as* $t\to+\infty$ *where*

$\chi=\left(\int\limits_\Omega a\,dx\right)^{-1}\int\limits_\Omega(ay_0+z_0)\,dx;$

that is,

$\lim\limits_{t\to\infty}\|(y(t),z(t))-(\chi,0)\|^2_\Upsilon=0.$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0,z_0)\neq\phi,\ \forall(y_0,z_0)\in\Upsilon$ and it is a compact set.

ii) $\omega(y_0,z_0)$ is invariant under the semi-group $S(t)$.

iii) Let $(y(t),z(t))=S(t)(y_0,z_0)$ be a solution of (3.8), then $\lim\limits_{t\to\infty}(y(t),z(t))\in\omega(y_0,z_0)$.

iv) $\omega(y_0,z_0)\subset D(\mathrm{T})$.

v) $t\to\|S(t)\omega\|^2_\Upsilon$ is a constant function, for any $(\omega_1,\omega_2)\in\omega(y_0,z_0)$.

We want to prove that $(y(t),z(t))\to(\chi,0)$, as t goes to ∞.

From (iii), it is sufficient to prove that $\omega(y_0,z_0)$ contains only elements of the form $(\chi,0)$.

Let $\omega_0\in\omega(y_0,z_0)$, we prove that $\omega_0=(\chi,0)$. We have

$\frac{d}{dt}\left(\|S(t)\omega_0\|^2_\Upsilon\right)=0\Rightarrow\left\langle\frac{d}{dt}\left(S(t)\omega_0\right),S(t)\omega_0\right\rangle_\Upsilon=0\Rightarrow\left\langle\frac{d}{dt}\omega(t),\omega(t)\right\rangle_\Upsilon=0$, where $\omega(t)=(y(t),z(t))$ is the solution of (3.8) corresponding to ω_0.

$\langle\mathrm{T}\omega(t),\omega(t)\rangle_\Upsilon=\int\limits_\Omega az^2dx=0$. But $a(x)\geq a_0>0$, thus $z=0$ on Ω.

Because $y_t=z=0$, then y is a constant with respect to t. Then $y_{tt}=0$

and $Ay = 0$ (from system (3.1)).

Therefore, using Green's formula

$-\int_\Omega Ay\, y\, dx = -\int_\Gamma \partial_A y\, y\, d\sigma + \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j y \partial_i y\, dx = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j y \partial_i y\, dx = 0$. But, $\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j y \partial_i y\, dx \geq \alpha_0 \int_\Omega \sum_{i=1}^n (\partial_i y)^2 dx$, where $\alpha_0 > 0$, therefore y is a constant with respect to x.

Finally, $y = \chi$ where χ is a constant.

Hence the ω-limit set contains only elements of the form $(\chi, 0)$, where χ is a constant, and we find $\lim_{t\to\infty}(y(t), z(t)) = (\chi, 0)$.

Now, we have to find the expression of χ. Because

$y_{tt}(x,t) - Ay(x,t) + a(x)\, y_t(x,t) = 0$ in $\Omega \times (0,\infty)$ and $\partial_A y = 0$ on Γ,

then $\left(\int_\Omega (y_t(x,t) + a(x)y(x,t))dx\right)' = \int_\Omega Ay\, dx = \int_\Gamma \partial_A y\, d\sigma = 0$, therefore $\int_\Omega (y_t(x,t) + a(x)y(x,t))dx$ is a constant function.

Thus,

$\int_\Omega (y_t(x,t) + a(x)y(x,t))dx = \int_\Omega (y_t(x,0) + a(x)y(x,0))dx = \int_\Omega (z_0 + ay_0)\, dx$,

$\forall\, t \in (0,\infty)$.

By passing to the limit where t goes to ∞, and using the fact that

$\lim_{t\to\infty}(y(t), z(t)) = (\chi, 0)$, we get $\int_\Omega (0 + a(x)\chi)\, dx = \int_\Omega (z_0 + ay_0)dx$, this implies that

$\chi \int_\Omega a\, dx = \int_\Omega (z_0 + ay_0)dx$,

so

$\chi = \left(\int_\Omega a\, dx\right)^{-1} \int_\Omega (z_0 + ay_0)dx.$ ∎

CHAPTER 4

THE WAVE EQUATION WITH DYNAMIC BOUNDARY CONDITIONS

4.1 Introduction

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. Consider the following wave equation with dynamic boundary conditions:

$$\begin{cases} y_{tt}(x,t) \,-\, Ay(x,t) \,+\, a(x)\, y_t(x,t) \,=0, & \text{in } \Omega\times(0,\infty) \\ m(x)y_{tt}(x,t)+\partial_A y(x,t)=0, & \text{on } \Gamma\times[0,\infty) \\ y(x,0) \,=\, y_0(x)\,, \quad y_t(x,0) =\, z_0(x) & \text{in } \Omega \\ y_t|_\Gamma(x,0)=w_0(x) & \text{on } \Gamma, \end{cases} \tag{4.1}$$

where $(y_0, z_0, w_0) \in H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)$ is given initial data, $A = \sum\limits_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$, $\partial_A = \sum\limits_{i,j=1}^{n} a_{ij}\nu_i\partial_j$, $\partial_k = \frac{\partial}{\partial x_k}$, $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω and $a_{ij} \in C^1(\overline{\Omega})$ with $a_{ij} = a_{ji}$, $\forall i, j = 1, ..., n$, and satisfying, for $\alpha_0 > 0$, $\sum\limits_{i,j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j \geq \alpha_0 \sum\limits_{i=1}^{n} \varepsilon_i^2$, $\forall\, (\varepsilon_1, ..., \varepsilon_n) \in \mathbb{R}^n$.

Moreover, there exist two positive constants a_0 and m_0 for which

$a(x) \in L^\infty(\Omega)$; $a(x) \geq a_0$, $a.e.x \in \Omega$.

$m(x) \in L^\infty(\Gamma)$; $m(x) \geq m_0$, $a.e.x \in \Gamma$.

4.2 Preliminaries and well-posedness of the problem

In this subsection, we study the existence and uniqueness of the solutions of the system (4.1). Let us consider the state space

$$\Upsilon_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma),$$

equipped with the inner product

$$\begin{aligned}\langle (y,z,w),(\tilde{y},\tilde{z},\tilde{w})\rangle_{\Upsilon_d} = & \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y\, \partial_j \tilde{y} + z\tilde{z} \right) dx + \int_\Gamma m\, w\, \tilde{w}\, d\sigma \\ & + \mu \left(\int_\Omega (z+ay)\, dx + \int_\Gamma m\, w\, d\sigma \right) \left(\int_\Omega (\tilde{z}+a\tilde{y})\, dx + \int_\Gamma m\, \tilde{w}\, d\sigma \right),\end{aligned} \tag{4.2}$$

where μ >0 is a constant to be determined. This inner product is inspired from the approach of [13] introduced for the boundary feedback case. The first result is stated in the following proposition.

Proposition 4.1 *The state space $\Upsilon_d = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma)$, endowed with the inner product (4.2) is a Hilbert space provided that μ is small enough.*

Proof. It is sufficient to show that the norm $\|\,.\,\|_{\Upsilon_d}$ induced by the inner product (4.2) is equivalent to the usual one $\|\,.\,\|_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}$; that is, we prove the existence of two positive constants K and $\tilde{K}$ such that

$$K\,\|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)} \le \|(y,z,w)\|^2_{\Upsilon_d} \le \tilde{K}\,\|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}\,. \tag{4.3}$$

On one hand,

$$\|(y,z,w)\|^2_{\Upsilon_d} = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i y \partial_j y\right) dx + \int_\Omega z^2 dx + \int_\Gamma m\,w^2\,d\sigma + \mu\left(\int_\Omega (z+ay)dx + \int_\Gamma m\,w\,d\sigma\right)^2.$$

Applying Hölder′s inequality and Young's inequality, we get

$$\|(y,z,w)\|_{\Upsilon_d} \le \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sup_{x\in\Omega}|a_{ij}(x)|\left((\partial_i y)^2 + (\partial_j y)^2\right) dx + \int_\Omega z^2 dx + \|m\|_\infty \int_\Gamma w^2 d\sigma + 4\mu\left(\int_\Omega z\,dx\right)^2 + 4\mu\left(\int_\Omega ay\,dx\right)^2 + 4\mu\left(\int_\Gamma mw\,d\sigma\right)^2$$

Let $a_1 = \max_{i,j} \sup_{x\in\Omega}|a_{ij}(x)|$, we have

$$\|(y,z,w)\|^2_{\Upsilon_d} \le na_1 \int_\Omega |\nabla y|^2 dx + \int_\Omega z^2 dx + \|m\|_\infty \int_\Gamma w^2\,d\sigma + 4\mu\,vol(\Omega)\int_\Omega z^2 dx + 4\mu\,\|a\|^2_\infty\,vol(\Omega)\int_\Omega y^2 dx + 4\mu\,\|m\|^2_\infty\,vol(\Gamma)\int_\Gamma w^2 d\sigma.$$

Therefore,

$$\|(y,z,w)\|^2_{\Upsilon_d} \leq na_1 \int_\Omega |\nabla y|^2 dx + 4\mu \|a\|^2_\infty vol(\Omega) \int_\Omega y^2 dx + (1+4\mu\, vol(\Omega)) \int_\Omega z^2 dx$$
$$+(\|m\|_\infty + 4\mu\ \|m\|^2_\infty\, vol(\Gamma)) \int_\Gamma w^2 d\sigma.$$

Let $\delta_0 = 4\mu\, vol(\Omega)\, \|a\|^2_\infty$, $\beta_0 = 1 + 4\mu\, vol(\Omega)$,

$\psi_0 = \|m\|_\infty + 4\mu \|m\|^2_\infty\ vol(\Gamma)$ and $\tilde{K} = \max\{na_1,\, \delta_0,\, \beta_0,\, \psi_0\}$.

Consequently,

$$\|(y,z,w)\|^2_{\Upsilon_d} \leq \tilde{K}\ \|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}\,. \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\|(y,z,w)\|^2_{\Upsilon_d} \geq\ & \alpha_0 \sum_{i=1}^n \int_\Omega (\partial_i y)^2\, dx + \int_\Omega z^2 dx\ + m_0 \int_\Gamma w^2\, d\sigma + \mu\left(\int_\Omega (z+ay)dx + \int_\Gamma mw\, d\sigma\right)^2 \\ \geq\ & \alpha_0 \int_\Omega |\nabla y|^2 dx + \int_\Omega z^2 dx +\ m_0 \int_\Gamma w^2 d\sigma +\ \mu\left(\int_\Omega ay\, dx\right)^2 \\ & +\mu\left(\int_\Omega z\, dx\ + \int_\Gamma mw\, d\sigma\right)^2 +\ 2\mu \left(\int_\Omega ay\, dx\right)\left(\int_\Omega z\, dx\ +\ \int_\Gamma mw\, d\sigma\right).\end{aligned} \tag{4.5}$$

Because

$$2\mu \left(\int_\Omega ay\, dx\right)\left(\int_\Omega z\, dx + \int_\Gamma mw\, d\sigma\right) \geq -\mu\left[\delta\left(\int_\Omega ay\, dx\right)^2 + \frac{1}{\delta}\left(\int_\Omega z\, dx + \int_\Gamma mw\, d\sigma\right)^2\right], \tag{4.6}$$

for any $\delta > 0$. Then, (4.5) and (4.6) imply that

$$\begin{aligned}\|(y,z,w)\|^2_{\Upsilon_d} \geq\ & \alpha_0 \int_\Omega |\nabla y|^2 dx +\ \int_\Omega z^2 dx +\ m_0 \int_\Gamma w^2 d\sigma +\ \mu(1-\delta)\left(\int_\Omega ay\, dx\right)^2 \\ & + \mu\left(1 - \tfrac{1}{\delta}\right)\left(\int_\Omega z\, dx + \int_\Gamma mw\, d\sigma\right)^2.\end{aligned} \tag{4.7}$$

Therefore, for any $0<\delta<1$ (so $1-\delta>0$ and $1-\frac{1}{\delta}<0$)

$$\mu\left(1-\tfrac{1}{\delta}\right)\left(\int_\Omega z\,dx+\int_\Gamma mw\,d\sigma\right)^2 \geq 2\mu\left(1-\tfrac{1}{\delta}\right)\left(\int_\Omega z\,dx\right)^2+2\mu\left(1-\tfrac{1}{\delta}\right)\left(\int_\Gamma mw\,d\sigma\right)^2$$
$$\geq 2\mu\left(1-\tfrac{1}{\delta}\right)\,vol(\Omega)\int_\Omega z^2\,dx$$
$$+\,2\mu\left(1-\tfrac{1}{\delta}\right)\|m\|_\infty^2\,vol(\Gamma)\int_\Gamma w^2 d\sigma,$$

hence, using (4.7) and (3.6) (given in Subsection 2.2) we get

$$\|(y,z,w)\|^2_{\Upsilon_d}\geq(\alpha_0-\mu(1-\delta))\int_\Omega|\nabla y|^2dx+\left(1+2\mu\left(1-\tfrac{1}{\delta}\right)vol(\Omega)\right)\int_\Omega z^2dx$$
$$+\left(m_0+2\mu\left(1-\tfrac{1}{\delta}\right)\|m\|^2_\infty vol(\Gamma)\right)\int_\Gamma w^2\,d\sigma+\tfrac{\mu(1-\delta)}{c_0}\int_\Omega y^2dx\,. \tag{4.8}$$

We choose $\mu>0$ and $0<\delta<1$ such that the coefficients of $\int_\Omega|\nabla y|^2dx$, $\int_\Omega y^2dx$, $\int_\Omega z^2dx$ and $\int_\Gamma w^2\,d\sigma$ are positive; that is,

$\alpha_0-\mu(1-\delta)>0$, which implies that $\mu<\frac{\alpha_0}{1-\delta}$.

$1+2\mu\left(1-\frac{1}{\delta}\right)vol(\Omega)>0$, then $\mu<\frac{1}{2\left(\frac{1}{\delta}-1\right)vol(\Omega)}$.

$m_0+2\mu\left(1-\frac{1}{\delta}\right)\|m\|^2_\infty\,vol(\Gamma)>0$, then $\mu<\frac{m_0}{2\mu\left(\frac{1}{\delta}-1\right)\|m\|^2_\infty vol(\Gamma)}$.

Because $0<\delta<1$, $\alpha_0>0$ and $m_0>0$, it is sufficient to choose $\mu>0$ such that

$0<\mu<\min\left\{\frac{\alpha_0}{1-\delta},\ \frac{1}{2\left(\frac{1}{\delta}-1\right)vol(\Omega)},\ \frac{m_0}{2\left(\frac{1}{\delta}-1\right)\|m\|^2_\infty vol(\Gamma)}\right\}$.

On the other hand, $c_0>0$, so $\frac{\mu(1-\delta)}{c_0}>0$.

Finally,

$$\|(y,z,w)\|^2_{\Upsilon_d}\geq K\,\|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}\,, \tag{4.9}$$

where $K=\min\left\{\alpha_0-\mu(1-\delta)\,,\ \frac{\mu(1-\delta)}{c_0},\ 1+2\mu\left(1-\frac{1}{\delta}\right)vol(\Omega),\ m_0+2\mu\left(1-\frac{1}{\delta}\right)\|m\|^2_\infty\,vol(\Gamma)\right\}$.

From (4.4) and (4.9), we get that

$$K \|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)} \leq \|(y,z,w)\|^2_{\Upsilon_d} \leq \tilde{K}\ \|(y,z,w)\|^2_{H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}.$$

So, the state space $\Upsilon_d = H^1(\Omega)\times L^2(\Omega)\times L^2(\Gamma)$, endowed with the inner product (4.2) is a Hilbert space. ∎

We turn now to the formulation of the system (4.1) in an abstract form in Υ_d. Let $z(t) = y_t(t)$, $w(t) = y_t(t)|_\Gamma$ and $\Phi(t) = (y(t), z(t), w(t))$. Then, the system (4.1) can be written as

$$\begin{cases} \Phi_t(t) + \mathrm{T}_d\Phi(t) = 0, \\ \Phi(0) = \Phi_0 = (y(0), z(0), w(0)) = (y_0, z_0, w_0), \end{cases} \tag{4.10}$$

where T_d is an unbounded linear operator defined by:

$$\mathrm{T}_d(y,z,w) = (-z, -Ay + az, \frac{1}{m(x)}\partial_A y), \qquad \forall (y,z,w) \in D(\mathrm{T}_d), \tag{4.11}$$

and

$$\begin{aligned} D(\mathrm{T}_d) &= \{(y,z,w) \in \Upsilon_d : \mathrm{T}_d(y,z,w) \in \Upsilon_d \text{ and } w = z \text{ on } \Gamma\} \\ &= \left\{(y,z,w) \in \Upsilon_d : (-z, -Ay + az, \tfrac{1}{m(x)}\partial_A y) \in \Upsilon_d \text{ and } w = z \text{ on } \Gamma\right\} \\ &= \left\{ \begin{array}{l} (y,z,w) \in \Upsilon_d : z \in H^1(\Omega),\ -Ay + az \in L^2(\Omega),\ \tfrac{1}{m(x)}\partial_A y \in L^2(\Gamma) \\ \text{and } w = z \text{ on } \Gamma \end{array} \right\} \\ &= \{(y,z,w) \in H^2(\Omega)\times H^1(\Omega)\times L^2(\Gamma),\ w = z \text{ on } \Gamma\}. \end{aligned} \tag{4.12}$$

We prove that T_d is maximal monotone operator.

We have, for any $(y,z,w) \in D(\mathrm{T}_d)$,

$$\begin{aligned}\langle \mathrm{T}_d(y,z,w),(y,z,w)\rangle_{\Upsilon_d} &= \left\langle (-z,\ -Ay+az,\ \tfrac{1}{m}\partial_A y),(y,z,w)\right\rangle_{\Upsilon_d}\\ &= \int_\Omega \left(\sum_{i,j=1}^n a_{ij}\partial_i(-z)\partial_j y\right) dx\ + \int_\Omega (-Ay+az)z\ dx + \int_\Gamma \partial_A y\, w\, d\sigma\\ &+\mu\left(\int_\Omega (-Ay+az-az)\,dx + \int_\Gamma \partial_A y\, d\sigma\right)\left(\int_\Omega (z+ay)\,dx + \int_\Gamma m\, w\, d\sigma\right).\end{aligned}$$

By applying the Green's formula and using the fact that $z = w$ on Γ, we get $\int_\Gamma \partial_A y\, w\, d\sigma - \int_\Gamma \partial_A y\, z\, d\sigma = \int_\Gamma \partial_A y\,(w-z)\, d\sigma = 0$. Then

$$\int_\Omega -Ay\, z\, dx + \int_\Gamma \partial_A y\, w\, d\sigma = -\int_\Gamma \partial_A y\, z\, d\sigma + \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i z\partial_j y\, dx + \int_\Gamma \partial_A y\, w\, d\sigma$$
$$= \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i z\partial_j y\, dx + \int_\Gamma \partial_A y\,(w-z)\, d\sigma = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_i z\partial_j y\, dx,$$

and

$$\int_\Omega (-Ay+az-\ az)\, dx\ +\ \int_\Gamma \partial_A y\, d\sigma\ =\ -\int_\Omega Ay\, dx\ +\ \int_\Gamma \partial_A y\, d\sigma\ =\ -\int_\Gamma \partial_A y\, d\sigma\ +\ \int_\Gamma \partial_A y\, d\sigma = 0.$$

So we conclude that $\langle \mathrm{T}_d(y,z,w),(y,z,w)\rangle_{\Upsilon_d} = \int_\Omega a\, z^2 dx \geq 0$, which means that T_d is monotone.

Now, we prove that $Id + \mathrm{T_d}$ is surjective.

Let $(f_1,f_2,f_3) \in \Upsilon_d$. We want to find $(y,z,w) \in D(T_d)$ such that

$(Id+\mathrm{T_d})(y,z,w) = (f_1,f_2,f_3)$; that is, $(y,z,w) + \mathrm{T}_d(y,z,w) = (f_1,f_2,f_3)$.

This means that

$$(y,z,w) + (-z,-Ay+az,\frac{1}{m}\partial_A y) = (f_1,f_2,f_3). \tag{4.13}$$

The first equation of (4.13) implies that $y - z = f_1$, so $z = y - f_1$.

The second equation of (4.13) becomes

$$-Ay + (a+1)z = f_2 \;\Leftrightarrow\; -Ay + (a+1)(y - f_1) = f_2,$$

which is equivalent to

$$-Ay + (a+1)y = f, \qquad \text{where } f = f_2 + (a+1)f_1 \;\in L^2(\Omega). \tag{4.14}$$

The third equation of (4.13) is reduced to $w + \frac{1}{m}\partial_A y = f_3$, thus $w = f_3 - \frac{1}{m}\partial_A y$. It is sufficient to prove that $-Ay + (a+1)y = f$ has a solution $y \in H^2(\Omega)$, and satisfying $\partial_A y = 0$ on Γ, therefore $z = y - f_1 \in H^1(\Omega)$, $w = f_3 - \frac{1}{m}\partial_A y \in L^2(\Gamma)$ and (4.13) holds.

By the variational formulation [4], let y be a solution of (4.14), then, for all $\varphi \in H^1(\Omega)$, $\int\limits_\Omega (a+1)\, y\varphi \, dx - \int\limits_\Omega Ay\, \varphi \, dx = \int\limits_\Omega f\varphi \, dx$, therefore

$$\int\limits_\Omega (a+1)y\varphi \, dx - \int\limits_\Gamma \partial_A y\, \varphi d\sigma + \int\limits_\Omega \sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi dx = \int\limits_\Omega f\varphi dx.$$

On the other hand, $w = z$ on Γ, this implies that $y - f_1 = f_3 - \frac{1}{m}\partial_A y$ on Γ, then

$$\tfrac{1}{m}\partial_A y + y = f_1 + f_3 \Leftrightarrow \partial_A y + my = m(f_1 + f_3) \text{ on } \Gamma.$$

Thus,

$$\int\limits_\Omega (a+1)y\varphi \, dx + \int\limits_\Gamma my\, \varphi d\sigma + \int\limits_\Omega \sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi dx = \int\limits_\Omega f\varphi dx + \int\limits_\Gamma m(f_1+f_3)\varphi d\sigma$$

Now, let us consider the application

$$\begin{aligned} F:\ & H^1(\Omega) \times H^1(\Omega) \to \mathbb{R} \\ & (y,\varphi) \quad \to F(y,\varphi) = \int\limits_\Omega \left((a+1)y\varphi + \sum_{i,j=1}^n a_{ij}\partial_i y \partial_j \varphi \right) dx + \int\limits_\Gamma my\varphi \, d\sigma \end{aligned}$$

It is clear that F is bilinear. We want to prove that F is continuous and coercive. We have

$$|F(y,\varphi)| = \left| \int\limits_{\Omega} \left((a+1)y\varphi + \sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi \right) dx + + \int\limits_{\Gamma} my\varphi \, d\sigma \right|$$
$$\leq \int\limits_{\Omega} \left((a+1)\,|y|\,|\varphi| + \left| \sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi \right| \right) dx + \int\limits_{\Gamma} m\,|y|\,|\varphi| \, d\sigma,$$

Applying Hölder′s inequality and putting $a_1 = \max_{i,j} \sup_{x\in\Omega} |a_{ij}(x)|$, we find

$$|F(y,z)| \leq (\|a\|_\infty + 1)\left(\int\limits_{\Omega} |y|^2 dx\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} |\varphi|^2 dx\right)^{\frac{1}{2}} + na_1 \left(\sum_{i=1}^{n} \int\limits_{\Omega} |\partial_i y|^2 dx\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \int\limits_{\Omega} |\partial_j \varphi|^2 dx\right)^{\frac{1}{2}}$$
$$+\|m\|_\infty \left(\int\limits_{\Gamma} |y|^2 d\sigma\right)^{\frac{1}{2}} \left(\int\limits_{\Gamma} |\varphi|^2 d\sigma\right)^{\frac{1}{2}}$$

$$\leq \left[(\|a\|_\infty + 1)\left(\int\limits_{\Omega} |y|^2 dx\right)^{\frac{1}{2}} + \left(\int\limits_{\Omega} |\nabla y|^2 dx\right)^{\frac{1}{2}} \right] \times \left[\left(\int\limits_{\Omega} |\varphi|^2 dx\right)^{\frac{1}{2}} + na_1 \left(\int\limits_{\Omega} |\nabla \varphi|^2 dx\right)^{\frac{1}{2}} \right]$$
$$+\|m\|_\infty \left[\left(\int\limits_{\Omega} \left(|y|^2 + |\nabla y|^2\right) dx\right)^{\frac{1}{2}} \right] \left[\left(\int\limits_{\Omega} \left(|\varphi|^2 + |\nabla \varphi|^2\right) dx\right)^{\frac{1}{2}} \right]$$

$$\leq \left[\sqrt{2}(\|a\|_\infty + 1)\left(\int\limits_{\Omega} \left(|y|^2 + |\nabla y|^2\right) dx\right)^{\frac{1}{2}} \right] \times \left[\sqrt{2}na_1 \left(\int\limits_{\Omega} \left(|\varphi|^2 + |\nabla \varphi|^2\right) dx\right)^{\frac{1}{2}} \right]$$
$$+\|m\|_\infty \left(\int\limits_{\Omega} \left(|y|^2 + |\nabla y|^2\right) dx\right)^{\frac{1}{2}} \left(\int\limits_{\Omega} \left(|\varphi|^2 + |\nabla \varphi|^2\right) dx\right)^{\frac{1}{2}}$$
$$= \left[2(\|a\|_\infty + 1)na_1 + \|m\|_\infty\right] \|y\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}.$$

Therefore,

$$|F(y,\varphi)| \leq c_1 \|y\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)}, \quad \text{where } c_1 = 2(\|a\|_\infty + 1)na_1 + \|m\|_\infty.$$

This implies that F is continuous.

On the other hand, we have

$$
\begin{aligned}
F(y,y) &= \int\limits_{\Omega}\left((a+1)|y|^2+\sum_{i,j=1}^{n}a_{ij}\partial_i y\partial_j y\right)dx\ +\int\limits_{\Gamma}m|y|^2d\sigma\\
&\geq \int\limits_{\Omega}\left((a_0+1)|y|^2+\alpha_0\sum_{i=1}^{n}\left(\partial_i y\right)^2\right)\\
&\geq \min\left\{(a_0+1),\alpha_0\right\}\left(\int\limits_{\Omega}\left(|y|^2+|\nabla y|^2\right)dx\right)\\
&= c_2\left\|y\right\|^2_{H^1(\Omega)}, \qquad \text{where } \mathrm{c}_2=\min\{(a_0+1),\alpha_0\}.
\end{aligned}
$$

This means F is coercive.

Let us consider the application

$$
\begin{aligned}
L:\ H^1(\Omega)\ &\to\ \mathbb{R}\\
\varphi\quad &\to\ L(\varphi)=\int\limits_{\Omega}f\varphi\,dx+\int\limits_{\Gamma}m(f_1+f_2)\varphi\,d\sigma.
\end{aligned}
$$

It is clear that L is linear. On the other hand, L is continuous, indeed,

$$
\begin{aligned}
|L(\varphi)| &= \left|\int\limits_{\Omega}f\varphi\,dx+\int\limits_{\Gamma}m\left(f_1+f_2\right)\varphi d\sigma\right| \leq \int\limits_{\Omega}|f|\,|\varphi|\,dx+\|m\|_{\infty}\int\limits_{\Gamma}|f_1+f_2|\,|\varphi|\,d\sigma\\
&\leq \left(\int\limits_{\Omega}|f|^2\right)^{\frac{1}{2}}\left(\int\limits_{\Omega}|\varphi|^2\right)^{\frac{1}{2}}+\|m\|_{\infty}\left(\int\limits_{\Gamma}|f_1+f_2|^2d\sigma\right)^{\frac{1}{2}}\left(\int\limits_{\Gamma}|\varphi|^2d\sigma\right)^{\frac{1}{2}}\\
&\leq \|f\|_{L^2(\Omega)}\left(\int\limits_{\Omega}(|\varphi|^2+|\nabla\varphi|^2)dx\right)^{\frac{1}{2}}+\|m\|_{\infty}\|f_1+f_2\|_{H^1(\Omega)}\,\|\varphi\|_{H^1(\Omega)}\\
&= \left[\|f\|_{L^1(\Omega)}+\|m\|_{\infty}\|f_1+f_2\|_{H^1(\Omega)}\right]\ \|y\|_{H^1(\Omega)}\,.
\end{aligned}
$$

Therefore,

$$
|L(\varphi)|\leq c_3\|\varphi\|_{H^1(\Omega)}, \qquad \text{where } c_3=\|f\|_{L^2(\Omega)}+\|m\|_{\infty}\|f_1+f_2\|_{H^1(\Omega)}.
$$

Applying Lax-Milgram theorem [4], we deduce that there exists a unique $y\in H^1(\Omega)$ such that $F(y,\varphi)=L(\varphi),\ \forall\varphi\in H^1(\Omega)$.

Then

$$
\int\limits_{\Omega}(a+1)y\varphi\,dx+\int\limits_{\Omega}\sum_{i,j=1}^{n}a_{ij}\partial_i y\partial_j\varphi\,dx\ +\int\limits_{\Gamma}my\varphi\,d\sigma=\int\limits_{\Omega}f\varphi\,dx\ +\int\limits_{\Gamma}m\left(f_1+f_3\right)\varphi d\sigma,
$$

$\forall\varphi\in H^1(\Omega)$,

therefore,

$$\int\limits_{\Omega} (a+1)y\varphi dx + \int\limits_{\Omega} \sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi dx + \int\limits_{\Gamma} my\varphi\, d\sigma - \int\limits_{\Omega} f\varphi dx - \int\limits_{\Gamma} m\,(f_1+f_3)\,\varphi d\sigma = 0,$$

$$\forall \varphi \in H^1(\Omega). \tag{4.15}$$

Applying Green's formula and taking $\varphi \in H_0^1(\Omega)$, we find

$$\int\limits_{\Omega} (a+1)y\varphi\, dx - \int\limits_{\Omega} Ay\,\varphi\, dx\; - \int\limits_{\Omega} f\varphi\, dx = 0\,, \qquad \forall \varphi \in H_0^1(\Omega);$$

that is,

$$\int\limits_{\Omega} ((a+1)y - Ay\; - f)\,\varphi\, dx = 0\,, \qquad \forall \varphi \in H_0^1(\Omega),$$

hence

$$\langle (a+1)y - Ay\; - f, \varphi\rangle_{L^2(\Omega)} = 0\,, \qquad \forall \varphi \in H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, we get $(a+1)y - Ay\; - f = 0$ in $L^2(\Omega)$, which is equivalent to $(a+1)y - Ay = f$ in $L^2(\Omega)$, so (4.14) holds.

Because $y \in H^1(\Omega)$ and $f \in L^2(\Omega)$, then from the elliptic regularity, we have $y \in H^2(\Omega)$.

Therefore, $z = y - f_1$ exists in $H^1(\Omega)$ and $w = f_3 - \frac{1}{m}\partial_A y$ exists in $L^2(\Gamma)$.

Now, we want to prove that $z = w$ on Γ, by going back to (4.15) and using Green's formula we get

$$\int\limits_{\Omega} [(a+1)y - Ay - f]\,\varphi dx + \int\limits_{\Gamma} [\partial_A y + my - m\,(f_1+f_3)]\,\varphi d\sigma = 0, \;\; \forall \varphi \in H^1(\Omega).$$

Therefore,

$$\int\limits_{\Gamma} \left[\partial_A y + my - m\left(f_1 + f_3\right)\right] \varphi d\sigma = 0 \qquad \forall \varphi \in H^1(\Omega).$$

Then $\partial_A y + my - m\left(f_1 + f_3\right)$ on Γ; that is, $\frac{1}{m}\partial_A y + y - \left(f_1 + f_3\right) = 0$ on Γ.

Hence, $y - f_1 = f_3 - \frac{1}{m}\partial_A y$ on Γ; that is, $z = w$ on Γ.

Therefore, we conclude that (4.13) has a unique solution $(y, z, w) \in D(\mathrm{T}_d)$; that is, $Id + \mathrm{T}_d$ is surjective. Finally, we deduce that T_d is maximal monotone.

By Hille-Yosida theorem (see [4], [34] and [36]), we get the following:

1) For all $\Phi_0 \in \Upsilon_d = H^1(\Omega) \cap L^2(\Omega) \cap L^2(\Gamma)$, there exists a unique $\Phi \in C(\mathbb{R}^+, \Upsilon_d)$ solution of (4.10). This implies that for any $(y_0, z_0, w_0) \in \Upsilon_0$ there exists a unique solution y of (4.1) satisfying

$y \in C(\mathbb{R}^+, H^2(\Omega))$, $y_t \in C(\mathbb{R}^+, L^2(\Omega))$, $y_t|_\Gamma \in C(\mathbb{R}^+, L^2(\Gamma))$

$\Leftrightarrow\ y \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^1(\Omega))$ and $y|_\Gamma \in C^1(\mathbb{R}^+, L^2(\Gamma))$.

2) If $\Phi_0 \in D(\mathrm{T}_d)$, then $\Phi \in C^1(\mathbb{R}^+, \Upsilon_d) \cap C(\mathbb{R}^+, D(\mathrm{T}_d))$; that is, for $(y_0, z_0, w_0) \in D(\mathrm{T}_d)$, $(y, z, w) \in C^1(\mathbb{R}^+, H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma)) \cap C(\mathbb{R}^+, H^2(\Omega) \times H^1(\Omega) \times L^2(\Gamma))$ solution of (4.1) satisfying

$y \in C^1(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega)$, $z \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^1(\Omega))$ and $w \in C^1(\mathbb{R}^+, L^2(\Gamma))$. Thus, $y \in C^2(\mathbb{R}^+, L^2(\Omega)) \cap C^1(\Re^+, H^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$ and $y|_\Gamma \in C^2(\mathbb{R}^+, L^2(\Gamma))$, since $z = y_t$ and $w = y_t|_\Gamma$.

4.3 Stabilization of the problem

In this subsection, we prove a stability result which is similar to the one obtained in [13] for the boundary feedback case.

Definition 4.1 *The ω-limit set is*

$$\omega(y_0, z_0, w_0) = \left\{ \begin{array}{l} (\omega_1, \omega_2, \omega_3) \in \Upsilon_d \ : \ \exists \{t_n\} \text{ an increasing sequence of positive numbers}; \\ \\ \lim\limits_{n\to\infty} \|(y(t_n), z(t_n), w(t_n)) - (\omega_1, \omega_2, \omega_3)\|_{\Upsilon_d} = 0 \end{array} \right\}.$$

Theorem 4.1 *For any initial data $\Phi_0 = (y_0, z_0, w_0) \in \Upsilon_d$, the solution*

$\Phi(t) = (y(t), z(t), w(t)) \ \to \ (\chi, 0, 0)$ *in* Υ_d *as* $t \to +\infty$, *where*

$$\chi = \left(\int_\Omega a\, dx\right)^{-1} \int_\Omega (ay_0 + z_0)\, dx;$$

that is,

$$\lim_{t\to\infty} \|(y(t), z(t), w(t)) - (\chi, 0, 0)\|^2_{\Upsilon_d} = 0.$$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0, z_0, w_0) \neq \phi, \ \ \forall (y_0, z_0, w_0) \in \Upsilon_d$ and it is a compact set.

ii) $\omega(y_0, z_0, w_0)$ is invariant under the semi – group $S(t) \quad (S(t)\omega(y_0, z_0, w_0) = \omega(y_0, z_0, w_0))$.

iii) Let $(y(t), z(t), w(t)) = S(t)(y_0, z_0, w_0)$ be a solution of (4.10), then

$\lim\limits_{t\to\infty} (y(t), z(t), w(t)) \in \omega(y_0, z_0, w_0)$.

iv) $\omega(y_0, z_0, w_0) \ \subset \ D(\mathrm{T}_d)$.

v) $t \ \to \ \|S(t)\omega\|^2_{\Upsilon_d}$ is a constant function, for any $(\omega_1, \omega_2, \omega_3) \in \omega(y_0, z_0, w_0)$.

We will prove that $(y(t), z(t), w(t))$ converges to $(\chi, 0, 0)$ as t goes to ∞. From (iii), it is sufficient to prove that $\omega(y_0, z_0, w_0)$ contains only elements of the form $(\chi, 0, 0)$.

Let $\omega_0 \in \omega(y_0, z_0, w_0)$, we prove that $\omega_0 = (\chi, 0, 0)$. We have

$$\frac{d}{dt}\left(\|S(t)\omega_0\|^2_{\Upsilon_d}\right) = 0 \Rightarrow \left\langle \frac{d}{dt}\left(S(t)\omega_0\right), S(t)\omega_0 \right\rangle_{\Upsilon_d} = 0 \Rightarrow \left\langle \frac{d}{dt}\omega(t), \omega(t) \right\rangle_{\Upsilon_d} = 0,$$

where $\omega(t) = (y(t), z(t), w(t))$ is the solution of (4.10) corresponding to ω_0.

$\langle \mathrm{T}_d\omega(t), \omega(t)\rangle_{\Upsilon_d} = \int\limits_\Omega az^2 dx = 0$. But, $a(x) \geq a_0 > 0$, so $z = 0$ on Ω.

Because $y_t = z = 0$, then y is a constant with respect to t. Therefore, $w = z|_\Gamma = 0$, so $y_{tt}|_\Gamma = 0$

Then $Ay = y_{tt} = 0$ and $\partial_A y|_\Gamma = 0$ (from system (4.1)).

Therefore, using Green's formula,

$-\int\limits_\Omega Ay\, y\, dx = -\int\limits_\Gamma \partial_A y\, y\, d\sigma + \int\limits_\Omega \sum\limits_{i,j=1}^n a_{ij}\partial_j y \partial_i y\, dx = \int\limits_\Omega \sum\limits_{i,j=1}^n a_{ij}\partial_j y \partial_i y\, dx = 0$. But, $\int\limits_\Omega \sum\limits_{i,j=1}^n a_{ij}\partial_j y \partial_i y\, dx \geq \alpha_0 \int\limits_\Omega \sum\limits_{i=1}^n (\partial_i y)^2 dx$, where $\alpha_0 > 0$, therefore y is a constant with respect to x.

Finally $y = \chi$, where χ is a constant.

Hence, the ω-limit set contains only elements of the form $(\chi, 0, 0)$, where χ is a constant, and we find $\lim\limits_{t\to\infty}(y(t), z(t), w(t)) = (\chi, 0, 0)$.

Now, we have to find the expression of χ. Because

$y_{tt}(x,t) - Ay(x,t) + a(x)\, y_t(x,t) = 0$ in $\Omega \times (0,\infty)$ and $\partial_A y = 0$ on $\Gamma \times (0,\infty)$, then $\left(\int\limits_\Omega (y_t(x,t) + a(x)y(x,t))dx\right)' = \int\limits_\Omega Ay\, dx = \int\limits_\Gamma \partial_A y\, d\sigma = 0$, therefore $\int\limits_\Omega (y_t(x,t) + a(x)y(x,t))dx$ is a constant function.

Thus,

$$\int\limits_\Omega (y_t(x,t) + a(x)y(x,t))dx = \int\limits_\Omega (y_t(x,0) + a(x)y(x,0))dx = \int\limits_\Omega (z_0 + ay_0)\, dx \quad \forall t \in (0,\infty).$$

By passing to the limit where t goes to ∞, and using the fact that

$\lim_{t\to\infty}(y(t), z(t)) = (\chi, 0)$, we get $\int_{\Omega}(0 + a(x)\chi)\,dx = \int_{\Omega}(z_0 + ay_0)dx$, this implies that

$$\chi \int_{\Omega} a\,dx = \int_{\Omega}(z_0 + ay_0)dx,$$

so

$$\chi = \left(\int_{\Omega} a\,dx\right)^{-1} \int_{\Omega}(z_0 + ay_0)dx. \qquad \blacksquare$$

CHAPTER 5

APPLICATIONS TO OTHER SYSTEMS

The method presented in the previous two chapters can be applied for a large class of distributed systems (where the classical energy defines only a semi-norm in the state space) to prove that the solution exists and converges to an equilibrium point (when the time goes to infinity). This equilibrium point can be determined explicitly in term of the parameters of the considered systems. We give here some particular applications to Petrovsky system, coupled wave-wave equations and elastic system. For more details concerning these systems, see [15]- [23] and the references therein. The proof of the obtained stability results of this chapter is inspired from the approach introduced in [13] for the case of boundary feedback.

5.1 Petrovsky system

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^4.

5.1.1 Static boundary conditions

We consider the following Petrovsky system with static boundary conditions:

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) + a(x) y_t(x,t) = 0 & \text{in } \Omega \times (0,\infty) \\ \partial_\nu y(x,t) = 0 & \text{on } \Gamma \times (0,\infty) \\ \partial_\nu \Delta y(x,t) = 0 & \text{on } \Gamma \times (0,\infty) \\ y(x,0) = y_0(x)\,,\ \ y_t(x,0) = z_0(x) & \text{in } \Omega\ , \end{cases} \tag{5.1}$$

where (y_0, z_0) is given initial datium in $V \times L^2(\Omega)$, where $V = \{\varphi \in H^2(\Omega);\, \partial_\nu \varphi = 0 \text{ on } \Gamma\}$, $a(x) \in L^\infty(\Omega)$, such that there exists $a_0 > 0$ satisfying $a(x) \geq a_0\ \forall x \in \Omega$, and $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω.

In this subsection, we study the existence and uniqueness of the solutions of the system (5.1). Let us consider the state space

$$\Upsilon_p = V \times L^2(\Omega),$$

equipped with the inner product

$$\langle (y,z), (\tilde{y}, \tilde{z}) \rangle_{\Upsilon_p} = \int_\Omega (\Delta y\, \Delta \tilde{y} + z\tilde{z}) dx + \varepsilon \left(\int_\Omega (z + ay) dx \right) \left(\int_\Omega (\tilde{z} + a\tilde{y}) dx \right), \tag{5.2}$$

where $\varepsilon > 0$ is a constant to be determined. The first result is stated in the following proposition.

Proposition 5.1 *The state space $\Upsilon_p = V \times L^2(\Omega)$ space endowed with the inner product (5.2) is a Hilbert space provided that ε is small enough.*

Proof. It is sufficient to show that the norm $\| \,.\, \|_{\Upsilon_p}$ induced by the inner product(5.2) is equivalent to the usual one $\| \,.\, \|_{H^2(\Omega)\times L^2(\Omega)}$; that is, we prove the existence of two positive constants K and $\tilde{K}$ such that

$$K \, \|(y,z)\|^2_{H^2(\Omega)\times L^2(\Omega)} \, \leq \, \|(y,z)\|^2_{\Upsilon_p} \, \leq \tilde{K} \, \|(y,z)\|^2_{H^2(\Omega)\times L^2(\Omega)} \, .$$

On one hand,

$$\|(y,z)\|^2_{\Upsilon_p} = \int\limits_{\Omega} |\Delta y|^2 dx \, + \int\limits_{\Omega} z^2 dx \, + \, \varepsilon \left(\int\limits_{\Omega} (z + ay) dx \right)^2 . \tag{5.3}$$

Applying Hölder inequality, we get

$\|(y,z)\|^2_{\Upsilon_p} \leq \int\limits_{\Omega} |\Delta y|^2 dx \, + \int\limits_{\Omega} z^2 dx \, + \, \varepsilon \, vol(\Omega) \int\limits_{\Omega} (|z| + |a| \, |y|)^2 dx.$

Using the fact that $2|a| \, |y| \, |z| \leq \|a\|_{\infty}(y^2 + z^2)$, we get

$$\begin{aligned}
\|(y,z)\|^2_{\Upsilon_p} &\leq \int\limits_{\Omega} |\Delta y|^2 dx + \int\limits_{\Omega} z^2 dx + \, \varepsilon vol(\Omega) \int\limits_{\Omega} z^2 dx + \varepsilon \, \|a\|^2_{\infty} \, vol(\Omega) \int\limits_{\Omega} y^2 dx \\
&\quad + \varepsilon \|a\|_{\infty} vol(\Omega) \int\limits_{\Omega} z^2 dx + \varepsilon \|a\|_{\infty} vol(\Omega) \int\limits_{\Omega} y^2 dx \\
&= \int\limits_{\Omega} |\Delta y|^2 dx \, + \, \varepsilon \|a\|_{\infty} vol(\Omega)(\|a\|_{\infty} + 1) \int\limits_{\Omega} y^2 dx \\
&\quad + (1 + \varepsilon \, vol(\Omega)(\|a\|_{\infty} + 1)) \int\limits_{\Omega} z^2 dx.
\end{aligned}$$

Let $\beta_0 = \varepsilon \|a\|_{\infty} vol(\Omega)(\|a\|_{\infty} + 1)$, $\eta_0 = 1 + \varepsilon \, vol(\Omega)(\|a\|_{\infty} + 1)$ and

$\tilde{K} = \max\{1,\ \beta_0,\ \eta_0\}$. Then

$$\|(y,z)\|^2_{\Upsilon_p} \leq \tilde{K}\ \|(y,z)\|^2_{H^2(\Omega)\times L^2(\Omega)}\,. \tag{5.4}$$

On the other hand,

$$\|(y,z)\|^2_{\Upsilon_p} = \int_\Omega |\Delta y|^2 dx + \int_\Omega z^2 dx + \varepsilon\left[\left(\int_\Omega z\,dx\right)^2 + \left(\int_\Omega ay\,dx\right)^2 + 2\left(\int_\Omega z\,dx\right)\left(\int_\Omega ay\,dx\right)\right].$$

But $2\varepsilon\left(\int_\Omega z\,dx\right)\left(\int_\Omega ay\,dx\right) \geq -\varepsilon\left[\delta\left(\int_\Omega ay\,dx\right)^2 + \frac{1}{\delta}\left(\int_\Omega z\,dx\right)^2\right],\quad \forall\delta > 0.$

Then

$$\|(y,z)\|^2_{\Upsilon_p} \geq \int_\Omega |\Delta y|^2 dx + \int_\Omega z^2 dx + \varepsilon(1-\delta)\left(\int_\Omega ay\,dx\right)^2 + \varepsilon\left(1-\frac{1}{\delta}\right)\left(\int_\Omega z\,dx\right)^2.$$

Using generalized Poincaré's inequality [3], we prove that there exists a positive constant c_0 such that

$$\int_\Omega y^2\,dx \leq c_0\left[\int_\Omega |\Delta y|^2 dx + \left(\int_\Omega ay\,dx\right)^2\right],\qquad \forall y \in V \tag{5.5}$$

which implies that

$$\left(\int_\Omega ay\,dx\right)^2 \geq \frac{1}{c_0}\int_\Omega y^2\,dx - \int_\Omega |\Delta y|^2 dx. \tag{5.6}$$

Therefore, for $0 < \delta < 1$ $\left(\text{so } 1-\frac{1}{\delta} < 0 \text{ and } 1-\delta > 0\right)$

$$\|(y,z)\|^2_{\Upsilon_p} \geq \int_\Omega |\Delta y|^2 dx + \int_\Omega z^2 dx + \frac{\varepsilon(1-\delta)}{c_0} \int_\Omega y^2 dx - \varepsilon(1-\delta) \int_\Omega |\Delta y|^2 dx$$
$$+\varepsilon \left(1 - \tfrac{1}{\delta}\right) vol(\Omega) \int_\Omega z^2 dx.$$

Consequently,

$$\|(y,z)\|^2_{\Upsilon_p} \geq (1-\varepsilon(1-\delta)) \int_\Omega |\Delta y|^2 dx + \frac{\varepsilon(1-\delta)}{c_0} \int_\Omega y^2 dx$$
$$+ \left(1 + \varepsilon \left(1 - \tfrac{1}{\delta}\right) vol(\Omega)\right) \int_\Omega z^2 dx.$$

We choose $\varepsilon > 0$ and $0 < \delta < 1$ such that the coefficients of $\int_\Omega |\Delta y|^2 dx$, $\int_\Omega y^2 dx$ and $\int_\Omega z^2 dx$ are positive; that is

$1 - \varepsilon(1-\delta) > 0$, which implies that $\varepsilon < \frac{1}{1-\delta}$.

Also, $1 + \varepsilon \left(1 - \frac{1}{\delta}\right) vol(\Omega) > 0$, so $\varepsilon < \frac{1}{\left(\frac{1}{\delta}-1\right) vol(\Omega)}$

Because, $0 < \delta < 1$, it is sufficient to choose $\varepsilon > 0$ such that

$0 < \varepsilon < \min \left\{ \frac{1}{1-\delta}, \frac{1}{\left(\frac{1}{\delta}-1\right) vol(\Omega)} \right\}$.

On the other hand, $c_0 > 0$, so $\frac{\varepsilon(1-\delta)}{c_0} > 0$.

Finally,

$$\|(y,z)\|^2_{\Upsilon_p} \geq K \|(y,z)\|^2_{H^2(\Omega)\times L^2(\Omega)}, \tag{5.7}$$

where $K = \min \left\{ 1 - \varepsilon(1-\delta), \frac{\varepsilon(1-\delta)}{c_0}, 1 + \varepsilon \left(1 - \frac{1}{\delta}\right) vol(\Omega) \right\}$.

From (5.4) and (5.7) we get that:

$$K \|(y,z)\|^2_{H^2(\Omega)\times L^2(\Omega)} \leq \|(y,z)\|^2_{\Upsilon_p} \leq \tilde{K} \|(y,z)\|^2_{H^2(\Omega)\times L^2(\Omega)}.$$

Therefore, the state space $\Upsilon_p = V \times L^2(\Omega)$ endowed with the inner product (5.2) is a Hilbert space. ▌

We turn now to the formulation of the system (5.1) in an abstract form in Υ_p [4]. Let $z(t) = y_t(t)$ and $\Phi(t) = (y(t), z(t))$. Then, the system (5.1) can be written as

$$
\begin{cases}
\Phi_t(t) + \mathrm{T}_p\Phi(t) = 0, \\
\\
\Phi(0) = \Phi_0 = (y(0), z(0)) = (y_0, z_0),
\end{cases}
\tag{5.8}
$$

where T_p is an unbounded linear operator defined by:

$$
\mathrm{T}_p(y,z) = (-z, \Delta^2 y + az), \qquad \forall (y,z) \in D(\mathrm{T}_p) \tag{5.9}
$$

and

$$
\begin{aligned}
D(\mathrm{T}_p) &= \{(y,z) \in \Upsilon_p : \ \mathrm{T}_p(y,z) \in \Upsilon_p \text{ and } \partial_\nu \Delta y = 0 \text{ on } \Gamma\} \\
&= \left\{ \begin{array}{l} (y,z) \in V \times L^2(\Omega) : \ (-z, \Delta^2 y + az) \in V \times L^2(\Omega) \\ \\ \text{and } \ \partial_\nu \Delta y = 0 \text{ on } \Gamma \end{array} \right\} \\
&= \left\{ \begin{array}{l} (y,z) \in V \times L^2(\Omega) : \ z \in V, \ \Delta^2 y + az \in L^2(\Omega) \\ \\ \text{and } \partial_\nu \Delta y = 0 \text{ on } \Gamma \end{array} \right\} \\
&= \left\{ \begin{array}{l} (y,z) : \ y \in V, \ \Delta^2 y \in L^2(\Omega), \ z \in V \\ \\ \ \text{and } \ \partial_\nu \Delta y = 0 \text{ on } \Gamma \end{array} \right\}. \\
&= \{(y,z) \in (H^4(\Omega) \cap V) \times V : \ \partial_\nu \Delta y = 0 \text{ on } \Gamma\}.
\end{aligned}
\tag{5.10}
$$

We prove that T_p is maximal monotone operator.

We have for any $(y,z) \in D(\mathrm{T}_p)$

$$
\begin{aligned}
\langle \mathrm{T}_p(y,z), (y,z) \rangle_{\Upsilon_p} &= \langle (-z, \Delta^2 y + az), (y,z) \rangle_{\Upsilon_p} \\
&= \int_\Omega \Delta(-z)\Delta y \, dx + \int_\Omega (\Delta^2 y + az) z \, dx \\
&\quad + \varepsilon \left(\int_\Omega (\Delta^2 y + az - az) dx \right) \left(\int_\Omega (z + ay) dx \right).
\end{aligned}
$$

By applying the Green's formula and the fact that $\partial_\nu \mathrm{y} = \partial_\nu \Delta y = 0$ on Γ, we find

$\int\limits_{\Omega} (\Delta^2 y + az - az)\, dx = \int\limits_{\Omega} \Delta^2 y\, dx \;=\; \int\limits_{\Gamma} \partial_\nu \Delta y\, d\sigma = 0$, and

$\int\limits_{\Omega} \Delta^2 y\, z\, dx \;=\; \int\limits_{\Gamma} \partial_\nu \Delta y\, z\, d\sigma \;-\; \int\limits_{\Omega} \nabla(\Delta y)\, \nabla z\; dx = \int\limits_{\Omega} \Delta z\, \Delta y\, dx - \int\limits_{\Gamma} \Delta y\, \partial_\nu z\, d\sigma$

$= \int\limits_{\Omega} \Delta z\, \Delta y\, dx.$

Then we get $\langle \mathrm{T}_p(y,z), (y,z)\rangle_{\Upsilon_p} \;=\; \int\limits_{\Omega} a\, z^2 dx \;\geq\; 0$, so we conclude that T_p is monotone.

Now, we prove that $Id + \mathrm{T}_p$ is surjective.

Let $(f_1, f_2) \in \Upsilon_p$. We want to find $(y,z) \in D(\mathrm{T}_p)$ such that

$(Id + \mathrm{T}_p)(y,z) = (f_1, f_2)$; that is, $(y,z) + \mathrm{T}_p(y,z) = (f_1, f_2)$.

This means that

$$(y,z) + (-z, \Delta^2 y + az) = (f_1, f_2). \tag{5.11}$$

The first equation of (5.11) implies that $y - z = f_1$, so $z = y - f_1$.

The second equation of (5.11) becomes

$$\Delta^2 y + (a+1)z = f_2 \Leftrightarrow \Delta^2 y + (a+1)(y - f_1) = f_2,$$

which is equivalent to

$$\Delta^2 y + (a+1)y = f, \qquad \text{where } f = (a+1)f_1 + f_2 \in L^2(\Omega). \tag{5.12}$$

By variational formulation [4], let y be a solution of(5.12) then $\forall \varphi \in V$,

$$\int\limits_{\Omega} \Delta^2 y\, \varphi dx + \int\limits_{\Omega} (a+1) y \varphi dx = \int\limits_{\Omega} f \varphi dx.$$

Applying Green's formula twice and using the fact that $\partial_\nu \Delta y = 0$ on Γ,we get

$\int\limits_{\Gamma} \partial_\nu \Delta y\, \varphi d\sigma - \int\limits_{\Omega} \nabla(\Delta y) \nabla \varphi dx + \int\limits_{\Omega} (a+1) y \varphi dx = \int\limits_{\Omega} f \varphi dx, \quad \forall \varphi \in V$; that is,

$\int\limits_{\Omega} \Delta y \Delta \varphi dx - \int\limits_{\Gamma} \partial_\nu \varphi \Delta y \, d\sigma + \int\limits_{\Omega} (a+1) y \varphi dx = \int\limits_{\Omega} f \varphi dx, \quad \forall \varphi \in V,$

hence

$\int\limits_{\Omega} \Delta y \Delta \varphi dx + \int\limits_{\Omega} (a+1) y \varphi dx = \int\limits_{\Omega} f \varphi dx, \quad \forall \varphi \in V$

Now, let us consider the application $F : V \times V \to \mathbb{R}$, defined by

$$F(y, \varphi) = \int\limits_{\Omega} \Delta y \Delta \varphi dx + \int\limits_{\Omega} (a+1) y \varphi dx,$$

which is bilinear and by using Hölder′s inequality we find that F is continuous and coercive.

Also, Let us consider the application $L : V \to \mathbb{R}$, defined by $L(\varphi) = \int\limits_{\Omega} f \varphi dx$, which is linear, and by using Hölder′s inequality we find that L is continuous.

As we did previously in the case of the wave equation with static boundary conditions (Subsection 2.2) and dynamic boundary conditions (Subsection 3.2), by using variational formulation and Lax-Millgram theorem [4], we conclude that (5.12) has a unique solution $y \in H^4(\Omega) \cap V$ satisfying $\partial_\nu \Delta y = 0$ on Γ, therefore $z = y - f_1 \in V$, and (5.12) holds.

We conclude that (5.11) has a unique solution $(y, z) \in D(\mathrm{T}_p)$; that is, $Id + \mathrm{T}_p$ is surjective. Finally, we conclude that $\mathrm{T_p}$ is maximal monotone operator.

By Hille-Yosida theorem (see [34] and [36]), we get the following:

1) For all $\Phi_0 \in \Upsilon_p = V \cap L^2(\Omega)$, there exists a unique $\Phi \in C(\mathbb{R}^+, \Upsilon_p)$ solution of (5.8). This implies that there exists a unique y is a solution of (5.1) satisfying $y \in C(\mathbb{R}^+, H^2(\Omega))$, $y_t \in C(\mathbb{R}^+, L^2(\Omega)) \Leftrightarrow y \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$.

2) If $\Phi_0 \in D(\mathrm{T}_p)$, then $\Phi \in C^1(\mathbb{R}^+, \Upsilon_p) \cap C(\mathbb{R}^+, D(\mathrm{T}_p))$; that is, for

$(y_0, z_0) \in D(\mathrm{T}_p)$, $(y, z) \in C^1(\mathbb{R}^+, H^2(\Omega) \times L^2(\Omega)) \cap C(\mathbb{R}^+, H^4(\Omega) \times H^2(\Omega))$

solution of (5.1), so

$y \in C^1(\mathbb{R}^+, H^2(\Omega)) \cap C(\mathbb{R}^+, H^4(\Omega))$, $z \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$

$\Leftrightarrow y \in C^2(\mathbb{R}^+, L^2(\Omega)) \cap C^1(\mathbb{R}^+, H^2(\Omega)) \cap C(\mathbb{R}^+, H^4(\Omega))$, since $z = y_t$.

Definition 5.1 *The ω-limit set is*

$$\omega(y_0, z_0) = \left\{ \begin{array}{l} (\omega_1, \omega_2) \in \Upsilon_p \, : \, \exists \{t_n\} \text{ an increasing sequence of positive numbers}\,; \\ \\ \lim\limits_{n \to \infty} \|(y(t_n), z(t_n)) - (\omega_1, \omega_2)\|_{\Upsilon_p} = 0 \end{array} \right\}.$$

Now, we prove the following stability result.

Theorem 5.1 *For any initial data $\Phi_0 = (y_0, z_0) \in \Upsilon_p$, the solution*

$\Phi(t) = (y(t), z(t)) \to (\chi, 0)$ *in* Υ_p *as* $t \to +\infty$ *where*

$$\chi = \left(\int_\Omega a \, dx \right)^{-1} \int_\Omega (a y_0 + z_0) \, dx;$$

that is,

$$\lim_{t \to \infty} \|(y(t), z(t)) - (\chi, 0)\|^2_{\Upsilon_p} = 0.$$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0, z_0) \neq \phi$, $\forall (y_0, z_0) \in \Upsilon_p$ and it is a compact set.

ii) $\omega(y_0, z_0)$ is invariant under the semi - group $S(t)$.

iii) Let $(y(t), z(t)) = S(t)(y_0, z_0)$ be a solution of (5.8), then

$\lim\limits_{t \to \infty} (y(t), z(t)) \in \omega(y_0, z_0)$.

iv) $\omega(y_0, z_0) \subset D(\mathrm{T}_p)$.

v) $t \to \|S(t)\omega\|^2_{\Upsilon_p}$ is a constant function, for any $(\omega_1, \omega_2) \in \omega(y_0, z_0)$.

We want to prove that $(y(t), z(t)) \to (\chi, 0)$, as t goes to ∞.

From (iii), it is sufficient to prove that $\omega(y_0, z_0)$ contains only elements of the form $(\chi, 0)$.

Let $\omega_0 \in \omega(y_0, z_0)$, we prove that $\omega_0 = (\chi, 0)$. We have

$\frac{d}{dt}\left(\|S(t)\omega_0\|^2_{\Upsilon_p}\right) = 0 \Rightarrow \left\langle \frac{d}{dt}(S(t)\omega_0)\,,\, S(t)\omega_0\right\rangle_{\Upsilon_p} = 0 \Rightarrow \left\langle \frac{d}{dt}\omega(t), \omega(t)\right\rangle_{\Upsilon_p} = 0,$

where $\omega(t) = (y(t), z(t))$ is the solution of (5.8) corresponding to ω_0.

$\langle \mathrm{T}_p\omega(t), \omega(t)\rangle_{\Upsilon_p} = \int\limits_\Omega az^2dx = 0$. But $a(x) \geq a_0 > 0$, thus, $z = 0$ on Ω.

Because $z = y_t = 0$ then y is a constant with respect to t.

Then $y_{tt} = 0$ and $\Delta^2 y = 0$ (from system (5.1)).

Therefore, using Green's formula two times, and using the fact that $\partial_\nu \mathrm{y} = \partial_\nu \Delta y = 0$ on Γ

$\int\limits_\Omega \Delta^2 y \times y\, dx = \int\limits_\Omega |\Delta y|^2 dx = 0$, which implies that y is a polynomial with degree ≤ 1 with respect to x; that is, there exist a constant τ and a constant vector $\lambda = (\lambda_1, ..., \lambda_n)$ satisfying

$y = \lambda.x + \tau$. Then $\partial_\nu y|_\Gamma = \lambda = 0$, therefore y is a constant with respect to x.

Finally, $y = \chi$, where χ is a constant.

Hence the ω-limit set contains only elements of the form $(\chi, 0)$, where χ is a constant, and we find $\lim\limits_{t\to\infty}(y(t), z(t)) = (\chi, 0)$.

Now, we have to find the expression of χ. Because,

$y_{tt}(x,t) + \Delta^2 y(x,t) + a(x)y_t(x,t) = 0$ in $\Omega \times (0,\infty)$ with $\partial_\nu y(x,t) = \partial_\nu \Delta y(x,t) = 0$ on Γ, then

$\left(\int\limits_\Omega ((y_t(x,t) + a(x)y(x,t))dx\right)' = -\int\limits_\Omega \Delta^2 y\, dx = 0$, therefore $\int\limits_\Omega ((y_t(x,t) + a(x)y(x,t))dx$ is a constant function. Thus,

$\int_\Omega (y_t(x,t) + a(x)y(x,t))dx = \int_\Omega (y_t(x,0) + a(x)y(x,0))dx = \int_\Omega (z_0 + ay_0)\, dx,$

$\forall t \in (0, \infty)$.

By passing to the limit where t goes to ∞, we get

$\int_\Omega (0 + \ a(x)\chi)\, dx = \int_\Omega (z_0 + ay_0)dx$, this implies that

$\chi \int_\Omega a\, dx = \int_\Omega (z_0 + ay_0)dx,$

so

$\chi = \left(\int_\Omega a(x)\, dx\right)^{-1} \int_\Omega (z_0 + ay_0)\, dx.$ ∎

5.1.2 Dynamic boundary conditions

We consider the following Petrovsky system with dynamic boundary conditions:

$$\begin{cases} y_{tt}(x,t) + \Delta^2 y(x,t) + a(x)y_t(x,t) = 0 & \text{in } \Omega \times (0,\infty) \\ \partial_\nu y(x,t) = 0 & \text{on } \Gamma \times (0,\infty) \\ -m(x)y_{tt}(x,t) + \partial_\nu \Delta y(x,t) = 0 & \text{on } \Gamma \times (0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x) & \text{in } \Omega \\ y_t|_\Gamma(x,0) = w_0(x) & \text{on } \Gamma, \end{cases} \tag{5.13}$$

where (y_0, z_0, w_0) is given initial datium in $V \times L^2(\Omega) \times L^2(\Gamma)$, where

$V = \{\varphi \in H^2(\Omega);\ \partial_\nu \varphi = 0 \text{ on } \Gamma\}$ and $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is the unit normal of Γ pointing towards the exterior of Ω.

Moreover, there exist two positive constants a_0 and m_0 for which

$a(x) \in L^\infty(\Omega);\ a(x) \geq a_0\, a.e.\ x \in \Omega$ and $m(x) \in L^\infty(\Omega);\ m(x) \geq m_0\ \ a.e.\ x \in \Gamma.$

First, we study the existence and uniqueness of the solutions of the system (5.13).

Let us consider the state space

$$\Upsilon_d = V \times L^2(\Omega) \times L^2(\Gamma),$$

equipped with the inner product

$$\begin{aligned}\langle (y,z,w),(\tilde{y},\tilde{z},\tilde{w})\rangle_{\Upsilon_d} &= \int_\Omega \Delta y \Delta \tilde{y} dx + \int_\Omega z\,\tilde{z}\,dx + \int_\Gamma m\,w\,\tilde{w}\,d\sigma \\ &+\mu\left(\int_\Omega (z+ay)dx + \int_\Gamma m\,w\,d\sigma\right)\left(\int_\Omega (\tilde{z}+a\tilde{y})dx + \int_\Gamma m\,\tilde{w}\,d\sigma\right),\end{aligned} \tag{5.14}$$

where μ >0 is a constant to be determined. The first result is stated in the following proposition:

Proposition 5.2 *The state space $\Upsilon_d = V \times L^2(\Omega) \times L^2(\Gamma)$ endowed with the inner product (5.14) is a Hilbert space provided that μ is small enough.*

Proof. It is sufficient to show that the norm $\|\,.\,\|_{\Upsilon_d}$ induced by the inner product (5.14) is equivalent to the usual one $\|\,.\,\|_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}$, that is, we prove the existence of two positive constants K and $\tilde{K}$ such that

$$K\,\|(y,z,w)\|^2_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)} \leq \|(y,z,w)\|^2_{\Upsilon_d} \leq \tilde{K}\,\|(y,z,w)\|^2_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}\,. \tag{5.15}$$

On one hand,

$$\|(y,z,w)\|^2_{\Upsilon_d} = \int_\Omega |\Delta y|^2 dx + \int_\Omega z^2 dx + \int_\Gamma m\,w^2 d\sigma + \mu\left(\int_\Omega (z+ay)dx + \int_\Gamma m\,w\,d\sigma\right)^2$$

Applying Hölder′s and Young's inequalities, we get

$$\|(y,z,w)\|^2_{\Upsilon_d} \leq \int\limits_\Omega |\Delta y|^2 dx + \int\limits_\Omega z^2 dx + \|m\|_\infty \int\limits_\Gamma w^2 d\sigma$$

$$+4\mu\left(\int\limits_\Omega z\,dx\right)^2 + 4\mu\left(\int\limits_\Omega ay\,dx\right)^2 + 4\mu\left(\int\limits_\Gamma mw\,d\sigma\right)^2.$$

Then,

$$\|(y,z,w)\|^2_{\Upsilon_d} \leq \int\limits_\Omega |\Delta y|^2 dx + \int\limits_\Omega z^2 dx + \|m\|_\infty \int\limits_\Gamma w^2 d\sigma + 4\mu\, vol(\Omega) \int\limits_\Omega z^2 dx$$

$$+ 4\mu\, \|a\|^2_\infty\, vol(\Omega) \int\limits_\Omega y^2 dx + 4\mu\, \|m\|^2_\infty\, vol(\Gamma) \int\limits_\Gamma w^2 d\sigma.$$

Therefore,

$$\|(y,z,w)\|^2_{\Upsilon_d} \leq \int\limits_\Omega |\Delta y|^2 dx + 4\mu\, \|a\|^2_\infty\, vol(\Omega) \int\limits_\Omega y^2 dx + (1 + 4\mu\, vol(\Omega)) \int\limits_\Omega z^2 dx$$

$$+\|m\|_\infty (1 + 4\mu \|m\|_\infty vol(\Gamma)) \int\limits_\Gamma w^2 d\sigma.$$

Let $\alpha_0 = 4\mu\, \|a\|^2_\infty\, vol(\Omega)$, $\beta_0 = 1 + 4\mu\, vol(\Omega)$, $\eta_0 = \|m\|_\infty (1 + 4\mu \|m\|_\infty vol(\Gamma))$ and $\tilde{K} = \max\{1,\, \alpha_0,\, \beta_0,\, \eta_0\}$, then

$$\|(y,z,w)\|^2_{\Upsilon_d} \leq \tilde{K}\, \|(y,z,w)\|^2_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}. \tag{5.16}$$

On the other hand, we have

$$\|(y,z,w)\|^2_{\Upsilon_d} \geq \int\limits_\Omega |\Delta y|^2 dx + \int\limits_\Omega z^2 dx + m_0 \int\limits_\Gamma w^2 d\sigma + \mu\left(\int\limits_\Omega ay dx\right)^2 +$$
$$+\mu\left(\int\limits_\Omega z\,dx + \int\limits_\Gamma mw\,d\sigma\right)^2 + 2\mu\left(\int\limits_\Omega ay dx\right)\left(\int\limits_\Omega z\,dx + \int\limits_\Gamma mw\,d\sigma\right). \tag{5.17}$$

But

$$2\mu\left(\int\limits_{\Omega} aydx\right)\left(\int\limits_{\Omega} z\,dx+\int\limits_{\Gamma} mw\,d\sigma\right)\geq -\mu\left[\delta\left(\int\limits_{\Omega} aydx\right)^2+\frac{1}{\delta}\left(\int\limits_{\Omega} z\,dx+\int\limits_{\Gamma} mw\,d\sigma\right)^2\right], \tag{5.18}$$

for any $\delta > 0$. Then, (5.17) and (5.18) imply that

$$\begin{aligned}\|(y,z,w)\|^2_{\Upsilon_d} \geq & \int\limits_{\Omega} |\Delta y|^2 dx + \int\limits_{\Omega} z^2 dx + m_0 \int\limits_{\Gamma} w^2 d\sigma \\ & + \mu(1-\delta)\left(\int\limits_{\Omega} aydx\right)^2 + \mu\left(1-\tfrac{1}{\delta}\right)\left(\int\limits_{\Omega} z\,dx + \int\limits_{\Gamma} mw\,d\sigma\right)^2.\end{aligned} \tag{5.19}$$

For $0 < \delta < 1$ (so $1-\delta > 0$ $1-\frac{1}{\delta} < 0$)

$$\begin{aligned}\mu\left(1-\tfrac{1}{\delta}\right)\left(\int\limits_{\Omega} z\,dx+\int\limits_{\Gamma} mw\,d\sigma\right)^2 &\geq 2\mu\left(1-\tfrac{1}{\delta}\right)\left(\int\limits_{\Omega} z\,dx\right)^2+2\mu\left(1-\tfrac{1}{\delta}\right)\left(\int\limits_{\Gamma} mw\,d\sigma\right)^2 \\ &\geq 2\mu\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\int\limits_{\Omega} z^2\,dx + 2\mu\left(1-\tfrac{1}{\delta}\right)\|m\|^2_\infty\, vol(\Gamma)\int\limits_{\Gamma} w^2\,d\sigma,\end{aligned} \tag{5.20}$$

hence, using (5.20) and (5.6) (given in Subsection 4.1.1) we get

$$\begin{aligned}\|(y,z,w)\|^2_{\Upsilon_d} \geq & \int\limits_{\Omega} |\Delta y|^2 dx + \int\limits_{\Omega} z^2 dx + m_0 \int\limits_{\Gamma} w^2 d\sigma + \tfrac{\mu(1-\delta)}{c_0}\int\limits_{\Omega} y^2 dx \\ & -\mu(1-\delta)\int\limits_{\Omega} |\Delta y|^2 dx + 2\mu\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\int\limits_{\Omega} z^2 dx \\ & +2\mu\left(1-\tfrac{1}{\delta}\right)\|m\|^2_\infty\, vol(\Gamma)\int\limits_{\Gamma} w^2 d\sigma.\end{aligned} \tag{5.21}$$

That is,

$$\|(y,z,w)\|_{\Upsilon_d}^2 \geq (1-\mu(1-\delta)) \int_\Omega |\Delta y|^2 dx + \frac{\mu(1-\delta)}{c_0} \int_\Omega y^2 dx$$
$$+ \left(1 + 2\mu\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\right) \int_\Omega z^2 dx + \left(m_0 + 2\mu\left(1-\tfrac{1}{\delta}\right) \|m\|_\infty^2 vol(\Gamma)\right) \int_\Gamma w^2 d\sigma. \tag{5.22}$$

We choose $\mu > 0$ and $0 < \delta < 1$ such that the coefficients of $\int_\Omega |\Delta y|^2 dx$, $\int_\Omega y^2 dx$, $\int_\Omega z^2 dx$ and $\int_\Gamma w^2 d\sigma$ are positive; that is,

$1-\mu(1-\delta) > 0$, which implies that $\mu < \frac{1}{1-\delta}$.

$1 + 2\mu\left(1-\frac{1}{\delta}\right) vol(\Omega) > 0$, then $\mu < \frac{1}{2\left(\frac{1}{\delta}-1\right) vol(\Omega)}$.

$m_0 + 2\mu\left(1-\frac{1}{\delta}\right) \|m\|_\infty^2 vol(\Gamma) > 0$, so $\mu < \frac{m_0}{2\left(\frac{1}{\delta}-1\right)\|m\|_\infty^2 vol(\Gamma)}$.

Because $0 < \delta < 1$ and $m_0 > 0$, it is sufficient to choose $\mu > 0$ such that

$0 < \mu < \min\left\{\frac{1}{1-\delta}, \frac{1}{2\left(\frac{1}{\delta}-1\right) vol(\Omega)}, \frac{m_0}{2\left(\frac{1}{\delta}-1\right)\|m\|_\infty^2 vol(\Gamma)}\right\}$.

On the other hand, $c_0 > 0$, so $\frac{\mu(1-\delta)}{c_0} > 0$.

Finally,

$$\|(y,z,w)\|_{\Upsilon_d}^2 \geq K \|(y,z,w)\|_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2, \tag{5.23}$$

where $K = \min\left\{1-\mu(1-\delta), \frac{\mu(1-\delta)}{c_0}, 1 + 2\mu\left(1-\frac{1}{\delta}\right) vol(\Omega), m_0 + 2\mu\left(1-\frac{1}{\delta}\right) \|m\|_\infty^2 vol(\Gamma)\right\}$.

From (5.16) and (5.23), we get that

$$K \|(y,z,w)\|_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2 \leq \|(y,z,w)\|_{\Upsilon_d}^2 \leq \tilde{K} \|(y,z,w)\|_{H^2(\Omega)\times L^2(\Omega)\times L^2(\Gamma)}^2.$$

Therefore, the state space $\Upsilon_d = V \times L^2(\Omega) \times L^2(\Gamma)$ endowed with the inner product (5.14) is a Hilbert space. ∎

We turn now to the formulation of the system (5.13) in an abstract form in Υ_d [4]. Let $z(t) = y_t(t)$, $w(t) = y_t(t)|_\Gamma$ and $\Phi(t) = (y(t), z(t), w(t))$. Then, the system (5.13) can be written as

$$\begin{cases} \Phi_t(t) + \mathrm{T}_d\Phi(t) = 0, \\ \Phi(0) = \Phi_0 = (y(0), z(0), w(0)) = (y_0, z_0, w_0), \end{cases} \tag{5.24}$$

where T_d is an unbounded linear operator defined by:

$$\mathrm{T}_d(y, z, w) = (-z, \Delta^2 y + az, -\frac{1}{m}\partial_\nu \Delta y) \tag{5.25}$$

and

$$\begin{aligned} D(\mathrm{T}_d) &= \left\{ (y, z, w) \in V \times L^2(\Omega) \times L^2(\Gamma) : \mathrm{T}_d(y, z, w) \in V \times L^2(\Omega) \times L^2(\Gamma) \text{ and } z|_\Gamma = w \right\} \\ &= \left\{ \begin{array}{l} (y, z, w) \in V \times L^2(\Omega) \times L^2(\Gamma) : \\ (-z,\ \Delta^2 y + az,\ -\frac{1}{m}\partial_\nu \Delta y) \in V \times L^2(\Omega) \times L^2(\Gamma) \text{ and } z|_\Gamma = w \end{array} \right\} \\ &= \left\{ \begin{array}{l} (y, z, w) \in V \times L^2(\Omega) \times L^2(\Gamma) : -z \in V, \Delta^2 y + az \in L^2(\Omega), \\ -\frac{1}{m}\partial_\nu \Delta y \in L^2(\Gamma) \text{ and } z|_\Gamma = w \end{array} \right\} \\ &= \left\{ (y, z, w) \in (H^4(\Omega) \cap V) \times V \times L^2(\Gamma) \text{ and } z|_\Gamma = w \right\}. \end{aligned} \tag{5.26}$$

We prove that T_d is maximal monotone operator.

We have for any $(y, z, w) \in D(\mathrm{T}_d)$

$$\begin{aligned} \langle \mathrm{T}_d(y, z, w), (y, z, w) \rangle_{\Upsilon_d} &= \left\langle (-z, \Delta^2 y + az, -\tfrac{1}{m}\partial_\nu \Delta y), (y, z, w) \right\rangle_{\Upsilon_d} \\ &= \int_\Omega \Delta(-z)\Delta y dx + \int_\Omega (\Delta^2 y + az) z \, dx + \int_\Gamma -\partial_\nu \Delta y \, w \, d\sigma \\ &+\mu \left(\int_\Omega (\Delta^2 y + az - az) dx + \int_\Gamma -\partial_\nu \Delta y \, d\sigma \right) \left(\int_\Omega (z + ay) dx + \int_\Gamma m \, w \, d\sigma \right). \end{aligned}$$

By applying Green formula and the fact that $\partial_\nu y = 0$ on Γ, and $z|_\Gamma = w$, we find

$$\int_{\Gamma} \partial_\nu \Delta y \, z \, d\sigma - \int_{\Gamma} \partial_\nu \Delta y \, w \, d\sigma = \int_{\Gamma} \partial_\nu \Delta y \, (z-w) \, d\sigma = 0,$$

$$\int_{\Omega} \Delta^2 y z \, dx - \int_{\Gamma} \partial_\nu \Delta y \, w \, d\sigma = \int_{\Gamma} \partial_\nu \Delta y \, z \, d\sigma = \int_{\Omega} \nabla(\Delta y) \nabla z \, dx - \int_{\Gamma} \partial_\nu \Delta y \, w \, d\sigma$$

$$= \int_{\Omega} \Delta z \, \Delta y \, dx - \int_{\Gamma} \Delta y \, \partial_\nu z \, d\sigma + \int_{\Gamma} \partial_\nu \Delta y (z-w) d\sigma = \int_{\Omega} \Delta z \Delta y \, dx,$$

$$\int_{\Omega} (\Delta^2 y + az - az) dx - \int_{\Gamma} \partial_\nu \Delta y \, d\sigma = \int_{\Omega} \Delta^2 y \, dx - \int_{\Gamma} \partial_\nu \Delta y \, d\sigma$$

$$= \int_{\Gamma} \partial_\nu \Delta y \, d\sigma - \int_{\Gamma} \partial_\nu \Delta y \, d\sigma = 0.$$

Then we get $\langle \mathrm{T}_d(y,z,w), (y,z,w) \rangle_{\Upsilon_d} = \int_{\Omega} a \, z^2 dx \geq 0$, so we conclude that $\mathrm{T_d}$ is monotone.

Now, we prove that $Id + \mathrm{T_d}$ is surjective.

Let $(f_1, f_2, f_3) \in \Upsilon_d$. We want to find $(y,z,w) \in D(T_d)$ such that

$(Id + \mathrm{T_d})(y,z,w) = (f_1, f_2, f_3)$; that is, $(y,z,w) + \mathrm{T}_d(y,z,w) = (f_1, f_2, f_3)$.

This means that

$$(y,z,w) + (-z, \Delta^2 y + az, -\frac{1}{m} \partial_\nu \Delta y) = (f_1, f_2, f_3). \tag{5.27}$$

The first equation of (5.27) implies that $y - z = f_1$, so $z = y - f_1$.

The second equation of (5.27) becomes

$$\Delta^2 y + (a+1) z = f_2 \Leftrightarrow \Delta^2 y + (a+1) y = (a+1) f_1 + f_2,$$

which is equivalent to

$$\Delta^2 y + (a+1) y = f, \qquad \text{where } f = (a+1) f_1 + f_2 \in L^2(\Omega) \tag{5.28}$$

The third equation is reduced to $w - \frac{1}{m}\partial_\nu \Delta y = f_3$, thus $w = \frac{1}{m}\partial_\nu \Delta y + f_3$.

By variational formulation, let y be a solution of (5.28), then $\forall \varphi \in V$,

$$\int_\Omega \Delta^2 y \, \varphi dx + \int_\Omega (a+1) y \varphi dx = \int_\Omega f \varphi dx$$

Applying Green's formula twice, we get

$\int_\Gamma \partial_\nu \Delta y \, \varphi d\sigma - \int_\Omega \nabla(\Delta y) \nabla \varphi dx + \int_\Omega (a+1) y \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in V$, hence

$\int_\Omega \Delta y \Delta \varphi dx - \int_\Gamma \partial_\nu \varphi \Delta y \, d\sigma + \int_\Gamma \partial_\nu \Delta y \, \varphi d\sigma + \int_\Omega (a+1) y \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in V,$

therefore

$\int_\Omega \Delta y \Delta \varphi dx + \int_\Gamma \partial_\nu \Delta y \, \varphi d\sigma + \int_\Omega (a+1) y \varphi dx = \int_\Omega f \varphi dx, \quad \forall \varphi \in V.$

On the other hand, $z = w$ on Γ, this implies that $y - f_1 = \frac{1}{m}\partial_\nu \Delta y + f_3$ on Γ,

then $\partial_\nu \Delta y = my - m(f_1 + f_3)$ on Γ. Thus

$\int_\Omega \Delta y \Delta \varphi dx + \int_\Gamma m y \varphi d\sigma + \int_\Omega (a+1) y \varphi dx = \int_\Omega f \varphi dx + \int_\Gamma m(f_1 + f_3) d\sigma, \quad \forall \varphi \in V$

Now, let us consider the application $F : V \times V \to \mathbb{R}$, defined by $F(y, \varphi) = \int_\Omega (a+1) y \varphi dx + \int_\Omega \Delta y \Delta \varphi dx + \int_\Gamma m y \varphi d\sigma$, which is bilinear, and by Hölder's inequality we find that F is continuous and coercive.

Also, let us consider the application $L : V \to \mathbb{R}$, defined by

$$L(\varphi) = \int_\Omega f \varphi dx + \int_\Gamma m(f_1 + f_3) d\sigma,$$

which is linear, and by Hölder's inequality we find that F is continuous.

As we did previously in the case of the wave equation with static boundary conditions (Subsection 2.2) and dynamic boundary conditions (Subsection 3.2), by using the variational formulation and Lax-Millgram theorem [4] we conclude that (5.28) has a unique solution $y \in H^4(\Omega) \cap V$, therefore $z = y - f_1$ exists in V and $w = \frac{1}{m}\partial_\nu \Delta y + f_3$ exists in $L^2(\Gamma)$.

We conclude that (5.27) has a unique solution $(y,z,w) \in D(\mathrm{T}_d)$; that is, $Id + \mathrm{T}_d$ is surjective.

By Hille-Yosida theorem (see [4], [34] and [36]), we get the following:

1) For all $\Phi_0 \in \Upsilon_d$, there exists a unique $\Phi \in C(\mathbb{R}^+, \Upsilon_d)$ solution of (5.24), this implies that there exists a unique y solution of (5.13) satisfying

$y \in C(\mathbb{R}^+, H^2(\Omega))$, $y_t \in C(\mathbb{R}^+, L^2(\Omega))$ and $y_t|_\Gamma \in C(\mathbb{R}^+, L^2(\Gamma))$. Thus

$y \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$ and $y|_\Gamma \in C^1(\mathbb{R}^+, L^2(\Gamma))$.

2) If $\Phi_0 \in D(\mathrm{T}_d)$ then $\Phi \in C^1(\mathbb{R}^+, \Upsilon_d) \cap C(\mathbb{R}^+, D(\mathrm{T}_d))$; that is, for $(y_0, z_0, w_0) \in D(\mathrm{T}_d)$, $(y,z,w) \in C^1(\mathbb{R}^+, H^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma)) \cap C(\mathbb{R}^+, H^4(\Omega) \times H^2(\Omega) \times L^2(\Gamma))$, solution of (5.13) satisfying

$y \in C^1(\mathbb{R}^+, H^2(\Omega)) \cap C(\mathbb{R}^+, H^4(\Omega))$, $z \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$, $w \in C^1(\mathbb{R}^+, L^2(\Gamma))$. Thus, $y \in C^2(\mathbb{R}^+, L^2(\Omega)) \cap C^1(\mathbb{R}^+, H^2(\Omega)) \cap C(\mathbb{R}^+, H^4(\Omega))$, and $y|_\Gamma \in C^2(\mathbb{R}^+, L^2(\Gamma)$, since $z = y_t$ and $w = y_t|_\Gamma$.

Definition 5.2 *The ω-limit set is*

$$\omega(y_0, z_0, w_0) = \left\{ \begin{array}{l} (\omega_1, \omega_2, \omega_3) \in \Upsilon_d \, : \, \exists \, \{t_n\} \text{ an increasing sequence of positive numbers}\,; \\ \\ \lim\limits_{n \to \infty} \|(y(t_n), z(t_n), w(t_n)) - (\omega_1, \omega_2, \omega_3)\|_{\Upsilon_d} = 0 \end{array} \right\}.$$

Now, we prove the following stability result.

Theorem 5.2 *For any initial data $\Phi_0 = (y_0, z_0, w_0) \in \Upsilon_d$, the solution $\Phi(t) = (y(t), z(t), w(t)) \to (\chi, 0, 0)$ in Υ_d as $t \to +\infty$, where*

$$\chi = \left(\int_\Omega a \, dx \right)^{-1} \int_\Omega (a y_0 + z_0) \, dx;$$

that is,

$$\lim_{t\to\infty} \|(y(t), z(t), w(t)) - (\chi, 0, 0)\|^2_{\Upsilon_d} = 0.$$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0, z_0, w_0) \neq \phi,\ \forall (y_0, z_0, w_0) \in \Upsilon_d$ and it is a compact set.

ii) $\omega(y_0, z_0, w_0)$ is invariant under the semi - group $S(t)$.

iii) Let $(y(t), z(t), w(t)) = S(t)(y_0, z_0, w_0)$ be a solution of (5.24), then

$\lim\limits_{t\to\infty} (y(t), z(t), w(t)) \in \omega(y_0, z_0, w_0)$.

iv) $\omega(y_0, z_0, w_0) \subset D(\mathrm{T}_d)$.

v) $t \rightarrow \|S(t)\omega\|^2_{\Upsilon_d}$ is a constant function, for any $(\omega_1, \omega_2, \omega_3) \in \omega(y_0, z_0, w_o)$.

We want to prove that $(y(t), z(t), w(t)) \to (\chi, 0, 0)$.

From (iii), it is sufficient to prove that $\omega(y_0, z_0, w_0)$ contains only elements of the form $(\chi, 0, 0)$.

Let $\omega_0 \in \omega(y_0, z_0, w_0)$, we prove that $\omega_0 = (\chi, 0, 0)$. We have

$\frac{d}{dt}\left(\|S(t)\omega_0\|^2_{\Upsilon_d}\right) = 0 \Rightarrow \left\langle \frac{d}{dt}\left(S(t)\omega_0\right), S(t)\omega_0 \right\rangle_{\Upsilon_d} = 0 \ \Rightarrow \left\langle \frac{d}{dt}\omega(t), \omega(t) \right\rangle_{\Upsilon_d} = 0,$

where $\omega(t) = (y(t), z(t), w(t))$ is the solution corresponding to ω_0

$\Rightarrow \langle \mathrm{T}\omega(t), \omega(t)\rangle_{\Upsilon_d} = 0 \Rightarrow \int\limits_{\Omega} a z^2 dx = 0$. But $a(x) \geq a_0 > 0$, thus $z = 0$ on Ω.

Because $w = z|_{\Gamma}$, then $w = 0$.

On the other hand, $z = y_t = 0$, implies that y is a constant with respect to t.

Then $y_{tt} = 0$ and $\Delta^2 y = 0$ (from system (5.13)).

Therefore, using Green's formula two times,

$$\int\limits_{\Omega} (\Delta^2 y)\, y\, dx = \int\limits_{\Gamma} y\,(\partial_\nu \Delta y)\, d\sigma - \int\limits_{\Omega} \nabla(\Delta y)\, \nabla y\, dx = \int\limits_{\Gamma} (\partial_\nu \Delta y)\, y\, d\sigma - \int\limits_{\Gamma} \Delta y\, \partial_\nu y\, d\sigma + \int\limits_{\Omega} |\Delta y|^2 dx$$

$$= \int\limits_{\Gamma} m y_{tt} y\, d\sigma - \int\limits_{\Gamma} \Delta y\, \partial_\nu y\, d\sigma + \int\limits_{\Omega} |\Delta y|^2 dx = \int\limits_{\Omega} |\Delta y|^2 dx = 0,$$

which implies that y is a polynomial with degree ≤ 1 with respect to x; that is, there exist a constant τ and a constant vector $\lambda = (\lambda_1, ..., \lambda_n)$ satisfying

$y = \lambda.x + \tau$. Then $\partial_\nu y|_\Gamma = \lambda = 0$, therefore y is a constant with respect to x.

Finally, $y = \chi$, where χ is a constant.

Hence the ω-limit set contains only elements of the form $(\chi, 0)$, where χ is a constant, and we find $\lim\limits_{t\to\infty}(y(t), z(t)) = (\chi, 0)$.

Now, we have to find the expression of χ. Because,

$y_{tt}(x,t) + \Delta^2 y(x,t) + a(x)y_t(x,t) = 0$ in $\Omega \times (0,\infty)$, then

$\left(\int\limits_\Omega ((y_t(x,t) + a(x)y(x,t))dx\right)' = -\int\limits_\Omega \Delta^2 y\, dx = 0$, therefore $\int\limits_\Omega ((y_t(x,t) + a(x)y(x,t))dx$ is a constant function. Thus

$\int\limits_\Omega (y_t(x,t) + a(x)y(x,t))dx = \int\limits_\Omega (y_t(x,0) + a(x)y(x,0))dx = \int\limits_\Omega (z_0 + ay_0)\, dx$

$\forall t \in (0,\infty)$.

By passing to the limit where t goes to ∞, we get

$\int\limits_\Omega (0 + \ a(x)\chi)\, dx = \int\limits_\Omega (z_0 + ay_0)dx$, this implies that

$\chi \int\limits_\Omega a\, dx = \int\limits_\Omega (z_0 + ay_0)dx$,

so $\chi = \left(\int\limits_\Omega a(x)\, dx\right)^{-1} \int\limits_\Omega (z_0 + ay_0)\, dx$. ∎

5.2 Coupled wave-wave equations

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. We consider the following coupled wave-wave system with static boundary conditions:

$$\begin{cases} y_{tt}(x,t) - Ay(x,t) + a_1 y_t(x,t) + bu_{tt} = 0 & \text{in } \Omega \times (0,\infty) \\ u_{tt}(x,t) - Bu(x,t) + a_2 u_t(x,t) + by_{tt} = 0 & \text{in } \Omega \times (0,\infty) \\ \partial_A y(x,t) = \partial_B u(x,t) = 0 & \text{on } \Gamma \times (0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x) & \text{in } \Omega \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{5.29}$$

where (y_0, u_0, z_0, v_0) is given initial data in $(H^1(\Omega))^2 \times (L^2(\Omega))^2$,
$A = \sum\limits_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$, $B = \sum\limits_{i,j=1}^{n} \partial_i(b_{ij}\partial_j)$ and $a_{ij}, b_{ij} \in C^1(\bar{\Omega})$ such that $a_{ij} = a_{ji}, b_{ij} = b_{ji}\, \forall i,j = 1,2,...,n$, and there exist $a_0, b_0 > 0$ satisfying
$\sum\limits_{i,j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j \geq a_0 \sum\limits_{i=1}^{n} \varepsilon_i^2\,,\ \sum\limits_{i=1}^{n} b_{ij}\varepsilon_i\varepsilon_j \geq b_0 \sum\limits_{i=1}^{n} \varepsilon_i^2 \quad \forall(\varepsilon_1, \varepsilon_2, ..., \varepsilon_n) \in \Re^n$.
Moreover, there exist two positive constants $a_{1,0}$ and $a_{2,0}$ for which

$a_1 \in L^\infty(\Omega)$; $a_1(x) \geq a_{1,0}$, *a.e.* $x \in \Omega$, $a_2 \in L^\infty(\Omega)$; $a_2(x) \geq a_{2,0}$, *a.e.* $x \in \Omega$

and $b \in L^\infty(\Omega)$ satisfies $\|b\|_\infty < 1$. For more details concerning these systems, see [20], [23] and the references therein.

5.2.1 Preliminaries and well-posedness of the problem

In this subsection, we study the existence and uniqueness of the solutions of the system (5.29). Let us consider the state space

$$\Upsilon_w = (H^1(\Omega))^2 \times (L^2(\Omega))^2,$$

equipped with the inner product

$$
\begin{aligned}
\langle (y,u,y_t,u_t),(\tilde{y},\tilde{u},\tilde{y}_t,\tilde{u}_t)\rangle_{\Upsilon_w} = & \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \tilde{y} + \sum_{i,j=1}^{n} b_{ij}\partial_i u \partial_j \tilde{u} \right) dx \\
& + \int_\Omega \left(y_t\tilde{y}_t + u_t\tilde{u}_t + b(y_t\tilde{u}_t + \tilde{y}_t u_t) \right) dx \\
& + \varepsilon \left(\int_\Omega (y_t + bu_t)dx + \int_\Omega a_1 y dx \right) \left(\int_\Omega (\tilde{y}_t + b\tilde{u}_t)dx + \int_\Omega a_1 \tilde{y} dx \right) \\
& + \varepsilon \left(\int_\Omega (by_t + u_t)dx + \int_\Omega a_2 u dx \right) \left(\int_\Omega (b\tilde{y}_t + \tilde{u}_t)dx + \int_\Omega a_2 \tilde{u}\, dx \right),
\end{aligned} \tag{5.30}
$$

where $\varepsilon > 0$ is a constant to be determined. The first result is stated in the following proposition:

Proposition 5.3 *The state space $\Upsilon_w = (H^1(\Omega))^2 \times (L^2(\Omega))^2$, endowed with the inner product (5.30) is a Hilbert space provided that ε is small enough.*

Proof. It is sufficient to show that the norm $\|\,.\,\|_{\Upsilon_w}$ induced by the inner product (5.30) is equivalent to the usual one $\|\,.\,\|_{(H^1(\Omega))^2\times(L^2(\Omega))^2}$; that is, we prove the existence of two positive constants K and $\tilde{K}$ such that

$$
K\,\|(y,u,y_t,u_t)\|^2_{(H^1(\Omega))^2\times(L^2(\Omega))^2} \leq \|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \leq \tilde{K}\,\|(y,u,y_t,u_t)\|^2_{(H^1(\Omega))^2\times(L^2(\Omega))^2} \tag{5.31}
$$

On one hand,

$$
\begin{aligned}
\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} = & \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j y + \sum_{i,j=1}^{n} b_{ij}\partial_i u \partial_j u \right) dx + \int_\Omega \left(y_t^2 + u_t^2 + 2b\, y_t u_t \right) dx \\
& + \varepsilon \left[\int_\Omega (y_t + bu_t)\, dx + \int_\Omega a_1 y\, dx \right]^2 + \varepsilon \left[\int_\Omega (by_t + u_t)\, dx + \int_\Omega a_2 u\, dx \right]^2.
\end{aligned}
$$

Applying Hölder′s and Young's inequalities, we get

$$
\begin{aligned}
\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \le{}& \tfrac{1}{2}\int_\Omega \sum_{i,j=1}^{n} \sup_{x\in\Omega}|a_{ij}(x)|\left((\partial_i y)^2+(\partial_j y)^2\right)dx\\
&+\tfrac{1}{2}\int_\Omega \sum_{i,j=1}^{n} \sup_{x\in\Omega}|b_{ij}(x)|\left((\partial_i u)^2+(\partial_j u)^2\right)dx \;+\int_\Omega \left(y_t^2+u_y^2+2\|b\|_\infty |y_t|\,|u_t|\right)dx\\
&+\varepsilon\;\left[\left(\int_\Omega (y_t+bu_t)\,dx\right)^2+\left(\int_\Omega a_1 y\,dx\right)^2+2\left(\int_\Omega (y_t+bu_t)\,dx\right)\left(\int_\Omega a_1 y\,dx\right)\right]\\
&+\varepsilon\;\left[\left(\int_\Omega (by_t+u_t)\,dx\right)^2+\left(\int_\Omega a_2 u\,dx\right)^2+2\left(\int_\Omega (by_t+u_t)\,dx\right)\left(\int_\Omega a_2 u\,dx\right)\right].
\end{aligned}
$$

Let $\tilde{a}=\max\limits_{i,j}\sup\limits_{x\in\Omega}|a_{ij}(x)|$, $\tilde{b}=\max\limits_{i,j}\sup\limits_{x\in\Omega}|b_{ij}(x)|$ and using the facts that
$2\left(\int_\Omega (y_t+bu_t)\,dx\right)\left(\int_\Omega a_1 y\,dx\right)\le\left(\int_\Omega (y_t+bu_t)\,dx\right)^2+\left(\int_\Omega a_1 y\,dx\right)^2$,
$2\left(\int_\Omega (by_t+u_t)\,dx\right)\left(\int_\Omega a_2 u\,dx\right)\le\left(\int_\Omega (by_t+u_t)\,dx\right)^2+\left(\int_\Omega a_2 u\,dx\right)^2$ and
$2\|b\|_\infty|y_t|\,|u_t|\le\|b\|_\infty(y_t^2+u_t^2)$, we get

$$
\begin{aligned}
\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \le{}& \tfrac{n\tilde{a}}{2}\int_\Omega \left(|\nabla y|^2+|\nabla y|^2\right)dx+\tfrac{n\tilde{b}}{2}\int_\Omega \left(|\nabla u|^2+|\nabla u|^2\right)dx\\
&+\int_\Omega \left(y_t^2+u_t^2+\|b\|_\infty\left(y_t^2+u_t^2\right)\right)dx\\
&+\varepsilon\;\left[2\left(\int_\Omega (y_t+bu_t)\,dx\right)^2+2\left(\int_\Omega a_1 y\,dx\right)^2\right]\\
&+\varepsilon\;\left[2\left(\int_\Omega (by_t+u_t)\,dx\right)^2+2\left(\int_\Omega a_2 u\,dx\right)^2\right],
\end{aligned}
$$

thus

$$
\begin{aligned}
\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \le{}& n\tilde{a}\int_\Omega |\nabla y|^2dx+n\tilde{b}\int_\Omega |\nabla u|^2dx+\int_\Omega |y_t|^2dx+(1+\|b\|_\infty)\int_\Omega (y_t^2+u_t^2)\,dx\\
&+\varepsilon\;\left[2vol(\Omega)\int_\Omega (y_t+bu_t)^2dx+2vol(\Omega)\,\|a_1\|^2_\infty\int_\Omega y^2\,dx\right]\\
&+\varepsilon\;\left[2vol(\Omega)\int_\Omega (by_t+u_t)^2dx+2vol(\Omega)\,\|a_2\|^2_\infty\int_\Omega u^2\,dx\right],
\end{aligned}
$$

then

$$\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \leq n\tilde{a}\int_\Omega |\nabla y|^2 dx + n\tilde{b}\int_\Omega |\nabla u|^2 dx + \int_\Omega |y_t|^2 dx + (1+\|b\|_\infty)\int_\Omega (y_t^2+u_t^2)\, dx$$
$$+2\varepsilon vol(\Omega)\ \left[\int_\Omega \left(y_t^2 + \|b\|^2_\infty u_t^2 + \|b\|_\infty (y_t^2+u_t^2)\right) dx + \|a_1\|^2_\infty \int_\Omega y^2\, dx\right]$$
$$+2\varepsilon vol(\Omega)\ \left[\int_\Omega \left(\|b\|^2_\infty y_t^2 + u_t^2 + \|b\|_\infty (y_t^2+u_t^2)\right) dx + \|a_2\|^2_\infty \int_\Omega u^2\, dx\right].$$

Therefore,

$$\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \leq n\tilde{a}\int_\Omega |\nabla y|^2 dx + n\tilde{b}\int_\Omega |\nabla u|^2 dx$$
$$+2\varepsilon\, vol(\Omega)\, \|a_1\|^2_\infty \int_\Omega y^2 dx\ + 2\varepsilon\, vol(\Omega)\, \|a_2\|^2_\infty \int_\Omega u^2 dx$$
$$+\left[1+\|b\|_\infty + 2\varepsilon\, vol(\Omega)\left(1+\|b\|^2_\infty + 2\|b\|_\infty\right)\right]\int_\Omega y_t^2\, dx$$
$$+\ \left[1+\|b\|_\infty + 2\varepsilon\, vol(\Omega)\left(1+\|b\|^2_\infty + 2\|b\|_\infty\right)\right]\int_\Omega u_t^2\, dx\,.$$

Let $\alpha_0 = 2\varepsilon vol(\Omega)\, \|a_1\|^2_\infty$, $\beta_0 = 2\varepsilon vol(\Omega)\, \|a_2\|^2_\infty$,

$\eta_0 = 1+\|b\|_\infty + 2\varepsilon\, vol(\Omega)\left(1+\|b\|^2_\infty + 2\|b\|_\infty\right)$ and $\tilde{K} = \max\left\{n\tilde{a}, n\tilde{b}, \alpha_0, \beta_0, \eta_0\right\}$.

Then

$$\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \leq \tilde{K}\, \|(y,u,y_t,u_t)\|^2_{(H^1(\Omega))^2\times(L^2(\Omega))^2}\,. \tag{5.32}$$

On the other hand,

$$\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \geq\ a_0\int_\Omega \sum_{i=1}^n (\partial_i y)^2 dx\ + b_0\int_\Omega \sum_{i=1}^n (\partial_i u)^2 dx + \int_\Omega y_t^2 dx$$
$$+\int_\Omega u_t^2 dx - \|b\|_\infty \int_\Omega (y_t^2+u_t^2)\, dx$$
$$+\varepsilon\ \left[\left(\int_\Omega (y_t+bu_t)\, dx\right)^2 + \left(\int_\Omega a_1 y\, dx\right)^2 + 2\left(\int_\Omega (y_t+bu_t)\, dx\right)\left(\int_\Omega a_1 y\, dx\right)\right]$$
$$+\varepsilon\ \left[\left(\int_\Omega (by_t+u_t)\, dx\right)^2 + \left(\int_\Omega a_2 y\, dx\right)^2 + 2\left(\int_\Omega (by_t+u_t)\, dx\right)\left(\int_\Omega a_2 y\, dx\right)\right].$$

But

$$2\varepsilon\left(\int_\Omega (y_t+bu_t)\, dx\right)\left(\int_\Omega a_1 y\, dx\right) \geq -\varepsilon\left[\delta\left(\int_\Omega a_1 y\, dx\right)^2 + \tfrac{1}{\delta}\left(\int_\Omega (y_t+bu_t)\, dx\right)^2\right]$$

and

$$2\varepsilon\left(\int_\Omega (by_t+u_t)\,dx\right)\left(\int_\Omega a_2u\,dx\right) \geq -\varepsilon\left[\delta\left(\int_\Omega a_2u\,dx\right)^2+\tfrac{1}{\delta}\left(\int_\Omega (by_t+u_t)\,dx\right)^2\right],$$

for all $\delta > 0$.

Then

$$\begin{aligned}\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \geq\ & a_0\int_\Omega |\nabla y|^2dx + b_0\int_\Omega |\nabla u|^2dx + (1-\|b\|_\infty)\int_\Omega (y_t^2+u_t^2)\,dx\\ &+\varepsilon(1-\delta)\left(\int_\Omega a_1y\,dx\right)^2+\varepsilon\left(1-\tfrac{1}{\delta}\right)\left(\int_\Omega (y_t+bu_t)\,dx\right)^2\\ &+\varepsilon(1-\delta)\left(\int_\Omega a_2u\,dx\right)^2+\varepsilon\left(1-\tfrac{1}{\delta}\right)\left(\int_\Omega (by_t+u_t)\,dx\right)^2.\end{aligned}$$

Using generalized Poincaré's inequality [3], we can prove that there exists two positive constants c_1 and c_2 such that $\int_\Omega y^2dx \leq c_1\left[\int_\Omega |\nabla y|^2dx+\left(\int_\Omega a_1y\,dx\right)^2\right]$ and $\int_\Omega u^2dx \leq c_2\left[\int_\Omega |\nabla u|^2dx+\left(\int_\Omega a_2u\,dx\right)^2\right]$, for any $(y,u)\in (H^1(\Omega))^2$ (see (3.6)). This implies that

$$\left(\int_\Omega a_1y\,dx\right)^2 \geq \tfrac{1}{c_1}\int_\Omega y^2dx-\int_\Omega |\nabla y|^2dx \text{ and } \left(\int_\Omega a_2u\,dx\right)^2 \geq \tfrac{1}{c_2}\int_\Omega u^2dx-\int_\Omega |\nabla u|^2dx.$$

Therefore, for any $0<\delta<1$ (so $1-\frac{1}{\delta}<0$ and $1-\delta>0$)

$$\begin{aligned}\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \geq\ & a_0\int_\Omega |\nabla y|^2dx + b_0\int_\Omega |\nabla u|^2dx + (1-\|b\|_\infty)\int_\Omega (y_t^2+u_t^2)\,dx\\ &+\tfrac{\varepsilon(1-\delta)}{c_1}\int_\Omega y^2dx-\varepsilon(1-\delta)\int_\Omega |\nabla y|^2dx\\ &+\varepsilon\left(1-\tfrac{1}{\delta}\right)vol(\Omega)\left(\int_\Omega \left(y_t^2+\|b\|_\infty^2u_t^2+\|b\|_\infty(y_t^2+u_t^2)\right)dx\right)\\ &+\tfrac{\varepsilon(1-\delta)}{c_2}\int_\Omega u^2dx-\varepsilon(1-\delta)\int_\Omega |\nabla u|^2dx\\ &+\varepsilon\left(1-\tfrac{1}{\delta}\right)vol(\Omega)\left(\int_\Omega \left(\|b\|_\infty^2y_t^2+u_t^2+\|b\|_\infty(y_t^2+u_t^2)\right)dx\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \geq\ & (a_0-\varepsilon(1-\delta))\int\limits_\Omega |\nabla y|^2dx + (b_0-\varepsilon(1-\delta))\int\limits_\Omega |\nabla u|^2dx \\ & +\tfrac{\varepsilon(1-\delta)}{c_1}\int\limits_\Omega y^2dx \ + \tfrac{\varepsilon(1-\delta)}{c_2}\int\limits_\Omega u^2dx \\ & + \left(1-\|b\|_\infty + \varepsilon\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\left[1+\|b\|^2_\infty + 2\|b\|_\infty\right]\right)\int\limits_\Omega y_t^2dx \\ & + \left(1-\|b\|_\infty + \varepsilon\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\left[1+\|b\|^2_\infty + 2\|b\|_\infty\right]\right)\int\limits_\Omega u_t^2dx\,.\end{aligned}$$

We choose $\varepsilon > 0$ and $0 < \delta < 1$ such that the coefficients of $\int\limits_\Omega |\nabla y|^2dx$, $\int\limits_\Omega |\nabla u|^2dx$, $\int\limits_\Omega y^2dx$, $\int\limits_\Omega u^2dx$, $\int\limits_\Omega y_t^2dx$ and $\int\limits_\Omega u_t^2dx$ are positive; that is

$a_0 - \varepsilon(1-\delta) > 0$, which implies that $\varepsilon < \frac{1-\delta}{a_0}$.

$b_0 - \varepsilon(1-\delta) > 0$, which implies that $\varepsilon < \frac{1-\delta}{b_0}$.

$1-\|b\|_\infty + \varepsilon\left(1-\frac{1}{\delta}\right) vol(\Omega)\left[1+\|b\|^2_\infty + 2\|b\|_\infty\right] > 0$, then

$0 < \varepsilon < \frac{1-\|b\|_\infty}{\left(\frac{1}{\delta}-1\right)vol(\Omega)\left[1+\|b\|^2_\infty+2\|b\|_\infty\right]}$.

Because $0<\delta<1$, $a_0>0$, $b_0>0$ and $\|b\|_\infty < 1$, it is sufficient to choose $\varepsilon > 0$ such that $0 < \varepsilon < \min\left\{\frac{1-\delta}{a_0},\ \frac{1-\delta}{b_0},\ \frac{1-\|b\|_\infty}{\left(\frac{1}{\delta}-1\right)vol(\Omega)\left[1+\|b\|^2_\infty+2\|b\|_\infty\right]}\right\}$.

On the other hand $c_1 > 0$ and $c_2 > 0$, so $\frac{\varepsilon(1-\delta)}{c_1} > 0$ and $\frac{\varepsilon(1-\delta)}{c_2} > 0$.

Finally

$$\|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \geq\ K\,\|(y,u,y_t,u_t)\|^2_{(H^1(\Omega))^2\times(L^2(\Omega))^2}\,, \tag{5.33}$$

where $K = \min\left\{\begin{array}{l} a_0-\varepsilon(1-\delta),\ b_0-\varepsilon(1-\delta),\ \frac{\varepsilon(1-\delta)}{c_1},\ \frac{\varepsilon(1-\delta)}{c_2}, \\ 1-\|b\|_\infty + \varepsilon\left(1-\frac{1}{\delta}\right) vol(\Omega)\left[1+\|b\|^2_\infty + 2\|b\|_\infty\right] \end{array}\right\}$.

From (5.32) and (5.33), we get that

$$K\,\|(y,u,y_t,u_t)\|^2_{(H^1(\Omega))^2\times(L^2(\Omega))^2} \leq \|(y,u,y_t,u_t)\|^2_{\Upsilon_w} \leq \tilde{K}\,\|(y,u,y_t,u_t)\|^2_{(H^1(\Omega))^2\times(L^2(\Omega))^2}\,.$$

Therefore, The state space $\Upsilon_w = (H^1(\Omega))^2 \times (L^2(\Omega))^2$ endowed with the inner product (5.30) is a Hilbert space. ∎

We turn now to the formulation of the system (5.29) in an abstract form in Υ_w. Let $z(t) = y_t(t)$, $v(t) = u_t(t)$ and $\Phi(t) = (y(t), u(t), z(t), v(t))$.

Then, the system (5.29) can be written as

$$\begin{cases} \Phi_t(t) + \mathrm{T}_w \Phi(t) = 0, \\ \Phi(0) = \Phi_0 = (y(0), u(0), z(0), v(0)) = (y_0, u_0, z_0, v_0), \end{cases} \tag{5.34}$$

where T_w is an unbounded linear operator defined by:

$$\mathrm{T}_w(y,u,z,v) = \left(-z, -v, \tfrac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right], \tfrac{1}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right]\right), \quad \forall (y,u,z,v) \in D(\mathrm{T}_w) \tag{5.35}$$

and

$$\begin{aligned} D(\mathrm{T}_w) &= \left\{ \begin{array}{l} (y,u,z,v) \in (H^1(\Omega))^2 \times (L^2(\Omega))^2 \;:\; \mathrm{T}_w(y,u,z,v) \in (H^1(\Omega))^2 \times (L^2(\Omega))^2 \\ \text{and } \partial_A y = \partial_B u = 0 \text{ on } \Gamma \end{array} \right\} \\ &= \left\{ \begin{array}{l} (y,u,z,v) \in (H^1(\Omega))^2 \times (L^2(\Omega))^2 \;: \\ -z \in H^1(\Omega), -v \in H^1(\Omega), \tfrac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] \in L^2(\Omega), \\ \tfrac{1}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] \in L^2(\Omega) \text{ and } \partial_A y = \partial_B u = 0 \text{ on } \Gamma \end{array} \right\} \end{aligned}$$

Now, $bBu - Ay \in L^2(\Omega)$ and $bAy - Bu \in L^2(\Omega)$. Because $b \in L^2(\Omega)$, then

$bBu - Ay \in L^2(\Omega)$ and $b^2 Ay - bBu \in L^2(\Omega)$, which implies that

$$(b^2 - 1)Ay \in L^2(\Omega). \tag{5.36}$$

Also, $b^2 Bu - bAy \in L^2(\Omega)$ and $bAy - Bu \in L^2(\Omega)$, which implies that

$$(b^2 - 1)Bu \in L^2(\Omega). \tag{5.37}$$

From (5.36) and (5.37) we get that $Ay \in L^2(\Omega)$ and $Bu \in L^2(\Omega)$, since $\|b\|_\infty^2 < 1$. Therefore,

$\Delta y \in L^2(\Omega)$, $\Delta u \in L^2(\Omega)$ and $(y, u) \in \left(H^1(\Omega)\right)^2$; that is, $(y, u) \in \left(H^2(\Omega)\right)^2$.

Finally,

$$D(\mathrm{T}_w) = \left\{(y, u, z, v) \in \left(H^2(\Omega)\right)^2 \times \left(H^1(\Omega)\right)^2 \;:\; \partial_A y = \partial_B u = 0 \text{ on } \Gamma\right\}. \tag{5.38}$$

We prove that T_w is maximal monotone operator.

We have, for any $(y, u, z, v) \in D(\mathrm{T}_w)$,

$\langle \mathrm{T}_w(y, u, z, v), (y, u, z, v)\rangle_{\Upsilon_w} =$

$\left\langle \left(-z,\, -v,\, \frac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right],\, \frac{1}{1-b^2}\left[bAy - Bu + ba_1 z + a_2 v\right]\right),\, (y, u, z, v)\right\rangle_{\Upsilon_w},$

therefore,

$$
\begin{aligned}
\langle \mathrm{T}_w(y,u,z,v),(y,u,z,v)\rangle_{\Upsilon_w} = &\int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}\partial_i(-z)\partial_j y + \sum_{i,j=1}^{n} b_{ij}\partial_i(-v)\partial_j u \right) dx \\
&+ \int_\Omega \tfrac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] z\,dx \\
&+ \int_\Omega \tfrac{1}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] v\,dx \\
&+ \int_\Omega \tfrac{b}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] v\,dx \\
&+ \int_\Omega \tfrac{b}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] z\,dx \\
&+\varepsilon \left(\begin{array}{l} \int_\Omega \tfrac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] dx \\ + \int_\Omega \tfrac{b}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] dx \\ + \int_\Omega a_1(-z)\,dx \end{array} \right) \\
&\left(\int_\Omega (z+bv)\,dx + \int_\Omega a_1 y\,dx \right) \\
&+\varepsilon \left(\begin{array}{l} \int_\Omega \tfrac{b}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] dx \\ + \int_\Omega \tfrac{1}{(1-b^2)}\left[bAy - Bu - ba_1 z + a_2 v\right] dx \\ + \int_\Omega a_2(-v)\,dx \end{array} \right) \\
&\left(\int_\Omega (bz+v)\,dx + \int_\Omega a_2 u\,dx \right).
\end{aligned}
$$

By applying the Green formula and using the fact that $\partial_A y = \partial_B u = 0$ on Γ, we find

$$\int_\Omega \tfrac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] z\, dx + \int_\Omega \tfrac{1}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] v\, dx +$$

$$\int_\Omega \tfrac{b}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] v\, dx + \int_\Omega \tfrac{b}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] z\, dx$$

$$= \int_\Omega \left(\tfrac{-Ayz}{1-b^2} + \tfrac{bBuz}{1-b^2} + \tfrac{a_1 z^2}{1-b^2} - \tfrac{ba_2 vz}{1-b^2}\right) dx + \int_\Omega \left(\tfrac{bAyv}{1-b^2} - \tfrac{Buv}{1-b^2} - \tfrac{ba_1 zv}{1-b^2} + \tfrac{a_2 v^2}{1-b^2}\right) dx +$$

$$\int_\Omega \left(\tfrac{-bAyv}{1-b^2} + \tfrac{b^2 Buv}{1-b^2} + \tfrac{ba_1 zv}{1-b^2} - \tfrac{b^2 a_2 v^2}{1-b^2}\right) dx + \int_\Omega \left(\tfrac{b^2 Ayz}{1-b^2} - \tfrac{bBuz}{1-b^2} - \tfrac{b^2 a_1 z^2}{1-b^2} + \tfrac{ba_2 vz}{1-b^2}\right) dx$$

$$= \int_\Omega \left(\tfrac{(b^2-1)Ayz}{1-b^2} + \tfrac{(b^2-1)Buv}{1-b^2} + \tfrac{(1-b^2)a_1 z^2}{1-b^2} + \tfrac{(1-b^2)a_2 v^2}{1-b^2}\right) dx$$

$$= \int_\Omega (-Ayz - Buv + a_1 z^2 + a_2 v^2) dx$$

$$= -\int_\Gamma \partial_A yz\, d\sigma + \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\partial_i z \partial_j y\, dx - \int_\Gamma \partial_B uv\, d\sigma + \int_\Omega \sum_{i,j=1}^{n} b_{ij}(x)\partial_i v \partial_j u\, dx$$

$$+ \int_\Omega (a_1 z^2 + a_2 v^2)\, dx$$

$$= + \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\partial_i z \partial_j y\, dx + \int_\Omega \sum_{i,j=1}^{n} b_{ij}(x)\partial_i v \partial_j u\, dx + \int_\Omega (a_1 z^2 + a_2 v^2)\, dx.$$

Also,

$$\int_\Omega \tfrac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] dx + \int_\Omega \tfrac{b}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] dx + \int_\Omega a_1(-z)\, dx$$

$$= \int_\Omega \left[\tfrac{-Ay}{1-b^2} + \tfrac{bBu}{1-b^2} + \tfrac{a_1 z}{1-b^2} - \tfrac{ba_2 v}{1-b^2}\right] dx + \int_\Omega \left[\tfrac{bAy}{1-b^2} - \tfrac{Bu}{1-b^2} - \tfrac{ba_1 z}{1-b^2} + \tfrac{a_2 v}{1-b^2}\right] dx + \int_\Omega a_1(-z)\, dx$$

$$= \int_\Omega \tfrac{(b^2-1)Ay}{1-b^2} dx + \int_\Omega \tfrac{(a_1 - b^2 a_1)z}{1-b^2} dx - \int_\Omega a_1 z\, dx = \int_\Omega (-Ay + a_1 z - a_1 z)\ dx$$

$$= -\int_\Omega Ay\, dx = -\int_\Gamma \partial_A y d\sigma = 0.$$

And

$$\int_\Omega \tfrac{b}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] dx + \int_\Omega \tfrac{1}{(1-b^2)}\left[bAy - Bu - ba_1 z + a_2 v\right] dx + \int_\Omega a_2(-v)\, dx$$

$$= \int_\Omega \left[\tfrac{-bAy}{1-b^2} + \tfrac{b^2 Bu}{1-b^2} + \tfrac{ba_1 z}{1-b^2} - \tfrac{b^2 a_2 v}{1-b^2}\right] dx + \int_\Omega \left[\tfrac{bAy}{1-b^2} - \tfrac{Bu}{1-b^2} - \tfrac{ba_1 z}{1-b^2} + \tfrac{a_2 v}{1-b^2}\right] dx + \int_\Omega a_2(-v)\, dx$$

$$= \int_\Omega \tfrac{(b^2-1)Bu}{1-b^2} dx + \int_\Omega \tfrac{(a_2 - b^2 a_2)v}{1-b^2} dx - \int_\Omega a_2 v\, dx = \int_\Omega (-Bu + a_2 v - a_2 v)\ dx$$

$$= -\int_\Omega Bu\, dx = -\int_\Gamma \partial_B u d\sigma = 0.$$

Then we get

$\langle \mathrm{T}_w(y,u,z,v),(y,u,z,v)\rangle_{\Upsilon_w} = \int\limits_{\Omega} (a_1 z^2 + a_2 v^2)\,dx \geq 0$, so we conclude that T_w is monotone.

Now, we prove that $Id + \mathrm{T}_w$ is surjective.

Let $(f_1, f_2, f_3, f_4) \in \Upsilon_w$. We want to find $(y,u,z,v) \in D(T_w)$ such that $(Id + \mathrm{T}_w)(y,u,z,v) = (f_1, f_2, f_3, f_4)$; that is,

$(y,u,z,v) + \mathrm{T}_w(y,u,z,v) = (f_1, f_2, f_3, f_4)$.

This means that

$$\begin{gathered}(y,u,z,v)+\\ \left(-z,-v,\tfrac{1}{1-b^2}\left[-Ay+bBu+a_1z-ba_2v\right],\tfrac{1}{1-b^2}\left[bAy-Bu-ba_1z+a_2v\right]\right)\\ =\ (f_1,f_2,f_3,f_4)\end{gathered} \tag{5.39}$$

The first equation of (5.39) implies that $y - z = f_1$, so $z = y - f_1$.

The second equation of (5.39) implies that $u - v = f_2$, so $v = u - f_2$.

The third equation of (5.39) becomes $z + \frac{1}{1-b^2}\left[-Ay + bBu + a_1 z - ba_2 v\right] = f_3$, which is equivalent to

$$-Ay + bBu + (1 - b^2 + a_1)y - ba_2 u = f_5, \tag{5.40}$$

where $f_5 = (1 - b^2 + a_1)f_1 - ba_2 f_2 + (1 - b^2)f_3 \ \in L^2(\Omega)$.

The fourth equation of (5.39) becomes $v + \frac{1}{1-b^2}\left[bAy - Bu - ba_1 z + a_2 v\right] = f_4$, which is equivalent to

$$bAy - Bu - ba_1 y + (1 - b^2 + a_2)u = f_6, \tag{5.41}$$

where $f_6 = -ba_1 f_1 + (1 - b^2 + a_2) f_2 + (1 - b^2) f_4 \in L^2(\Omega)$.

Multiplying (5.41) by b and adding to (5.40) we get

$(b^2 - 1)Ay + (1 - b^2 + a_1 - b^2 a_1)y + (b - b^3 + ba_2 - ba_2)u = f_5 + bf_6,$

which is equivalent to

$$-Ay + (1 + a_1)y + bu = f_7, \tag{5.42}$$

where $f_7 = \frac{f_5 + bf_6}{1 - b^2} \in L^2(\Omega)$.

Similarly, multiplying (5.40) by b and adding to (5.41) we get

$(b^2 - 1)Bu + (b - b^3 + a_1 b - a_1 b)y + (1 - b^2 + a_2 - b^2 a_2)u = bf_5 + f_6,$

which is equivalent to

$$-Bu + by + (1 + a_2)u = f_8, \tag{5.43}$$

where $f_8 = \frac{bf_5 + f_6}{1 - b^2} \in L^2(\Omega)$.

So we have to prove that the system

$$\begin{cases} -Ay + (1 + a_1)y + bu = f_7 \\ -Bu + by + (1 + a_2)u = f_8 \end{cases} \tag{5.44}$$

has a unique solution $(y, u) \in \left(H^2(\Omega)\right)^2$ satisfying $\partial_A y = \partial_B u = 0$ on Γ.

Let (y, u) be a solution of (5.44); then, for any $(\varphi, \psi) \in \left(H^1(\Omega)\right)^2$, we have

$$\int\limits_\Omega (-Ay + (1 + a_1)y + bu)\, \varphi \, dx + \int\limits_\Omega (-Bu + by + (1 + a_2)u)\, \psi \, dx = \int\limits_\Omega (f_7 \varphi + f_8 \psi)\, dx.$$

Using the Green's formula and the fact that $\partial_A y = \partial_B u = 0$ on Γ, we get

$$\int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi + (1+a_1)y\varphi + bu\varphi \right) dx + \int_{\Omega} \left(\sum_{i,j=1}^{n} b_{ij}\partial_i u \partial_j \psi + by\psi + (1+a_2)u\psi \right) dx$$
$$= \int_{\Omega} (f_7\varphi + f_8\psi)\, dx.$$

Now, let us consider the application $F : (H^1(\Omega))^2 \times (H^1(\Omega))^2 \to \mathbb{R}$, defined by

$$\begin{aligned} F((y,u),(\varphi,\psi)) = & \int_{\Omega} \left(\sum_{i,j=1}^{n} a_{ij}\partial_i y \partial_j \varphi + (1+a_1)y\varphi + bu\varphi \right) dx \\ & + \int_{\Omega} \left(\sum_{i,j=1}^{n} b_{ij}\partial_i u \partial_j \psi + by\psi + (1+a_2)u\psi \right) dx, \end{aligned}$$

which is bilinear, and by using Hölder′s inequality we find that F is continuous and coercive.

Also, let us consider the application $L : (H^1(\Omega))^2 \to \mathbb{R}$, defined by

$L(\varphi,\psi) = \int_{\Omega} (f_7\varphi + f_8\psi)\, dx$, which is linear and, by using Hölder′s inequality, we find that L is continuous.

As we did previously in the case of the wave equation with static boundary conditions (Subsection 2.2) and dynamic boundary conditions (Subsection 3.2), by using variational formulation and Lax-Millgram theorem [4], we conclude that (5.44) has a unique solution $(y,u) \in (H^2(\Omega))^2$ and satisfying $\partial_A y = \partial_B u = 0$ on Γ, therefore $z = y - f_1 \in H^1(\Omega)$, $v = u - f_2 \in H^1(\Omega)$ and (5.44) holds.

We conclude that (5.39) has a unique solution $(y,u,z,v) \in D(\mathrm{T}_w)$; that is $Id + \mathrm{T}_w$ is surjective. Finally, we conclude that T_w is maximal monotone operator.

By Hille-Yosida theorem (see [4], [27], [34] and [36]), we get the following:

1) For all $\Phi_0 \in \Upsilon_w = (H^1(\Omega))^2 \times (L^2(\Omega))^2$, there exists a unique $\Phi \in C(\mathbb{R}^+, \Upsilon_w)$ solution of (5.34). This implies that there exists a unique solution (y, u) of (5.29) satisfying $y \in C(\mathbb{R}^+, H^1(\Omega))$, $y_t \in C(\mathbb{R}^+, L^2(\Omega))$, $u \in C(\mathbb{R}^+, H^1(\Omega))$,

$u_t \in C(\mathbb{R}^+, L^2(\Omega)) \;\Leftrightarrow\; (y, u) \in C^1\left(\mathbb{R}^+, (L^2(\Omega))^2\right) \cap C\left(\mathbb{R}^+, (H^1(\Omega))^2\right)$.

2) If $\Phi_0 \in D(\mathrm{T}_w)$, then $\Phi \in C^1(\mathbb{R}^+, \Upsilon_w) \cap C(\mathbb{R}^+, D(\mathrm{T}_w))$; that is, for $(y_0, u_0, z_0, v_0) \in D(\mathrm{T}_w)$,

$(y, u, z, v) \in C^1(\mathbb{R}^+, (H^1(\Omega))^2 \times (L^2(\Omega))^2) \cap C(\mathbb{R}^+, (H^2(\Omega))^2 \times (H^1(\Omega))^2)$ is a unique solution of (5.29) satisfying

$y \in C^1(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$, $z \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^1(\Omega))$,

$u \in C^1(\mathbb{R}^+, H^1(\Omega)) \cap C(\mathbb{R}^+, H^2(\Omega))$ and $v \in C^1(\mathbb{R}^+, L^2(\Omega)) \cap C(\mathbb{R}^+, H^1(\Omega))$.

Thus, $(y, u) \in C^2\left(\mathbb{R}^+, (L^2(\Omega))^2\right) \cap C^1\left(\mathbb{R}^+, (H^1(\Omega))^2\right) \cap C\left(\mathbb{R}^+, (H^2(\Omega))^2\right)$,

since $z = y_t$ and $v = u_t$.

5.2.2 Stabilization of the problem

In this subsection, we prove the following stability result.

Definition 5.3 *The ω-limit set is*

$$\omega(y_0, u_0, z_0, v_0) = \left\{ \begin{array}{l} (\omega_1, \omega_2, \omega_3, \omega_4) \in \Upsilon_w \,:\, \exists \{t_n\} \text{ an increasing sequence of positive} \\ \text{numbers};\ \lim\limits_{n\to\infty} \|(y(t_n), u(t_n), z(t_n), v(t_n)) + (\omega_1, \omega_2, \omega_3, \omega_4)\|_{\Upsilon_w} = 0 \end{array} \right\}.$$

Theorem 5.3 *For any initial data* $\Phi_0 = (y_0, u_0, z_0, v_0) \in \Upsilon_w$, *the solution*

$\Phi(t) = (y(t), u(t), z(t), v(t)) \to (\chi_1, \chi_2, 0, 0)$ *in* Υ_w *as* $t \to \infty$ *where*

$\chi_1 \left(\int\limits_\Omega a_1\, dx \right) + \chi_2 \left(\int\limits_\Omega a_2\, dx \right) = \left[\int\limits_\Omega (1 + b)(z_0 + v_0) dx + \int\limits_\Omega (a_1 y_0 + a_2 u_0) dx \right]$;

that is,

$$\lim_{t\to\infty} \|(y(t), u(t), z(t), v(t)) - (\chi_1, \chi_2, 0, 0)\|^2_{\Upsilon_w} = 0$$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y_0, u_0, z_0, v_0) \neq \phi$, $\forall (y_0, u_0, z_0, v_0) \in \Upsilon_w$ and it is compact set.

ii) $\omega(y_0, u_0, z_0, v_0)$ is invariant under the semi-group $S(t)$.

iii) Let $(y(t), u(t), z(t), v(t)) = S(t)(y_0, u_0, z_0, v_0)$ be a solution of (5.34), then $\lim\limits_{t\to\infty} (y(t), u(t), z(t), v(t)) \in \omega(y_0, u_0, z_0, v_0)$.

iv) $\omega(y_0, u_0, z_0, v_0) \subset D(\mathrm{T}_w)$.

v) $t \to \|S(t)\omega\|^2_{\Upsilon_w}$ is a constant function, for any $(\omega_1, \omega_2, \omega_3, \omega_4) \in \omega(y_0, u_0, z_0, v_0)$.

We want to prove that $(y(t), u(t), z(t), v(t)) \to (\chi_1, \chi_2, 0, 0)$, as t goes to ∞.

From (iii), it is sufficient to prove that $\omega(y_0, u_0, z_0, v_0)$ contains only elements of the form $(\chi_1, \chi_2, 0, 0)$.

Let $\omega_0 \in \omega(y_0, u_0, z_0, v_0)$, we prove that $\omega = (\chi_1, \chi_2, 0, 0)$. We have

$\frac{d}{dt}\left(\|S(t)\omega_0\|^2_{\Upsilon_w}\right) = 0 \Rightarrow \left\langle \frac{d}{dt}\left(S(t)\omega_0\right), S(t)\omega_0 \right\rangle_{\Upsilon_w} = 0 \Rightarrow \left\langle \frac{d}{dt}\omega(t), \omega(t) \right\rangle_{\Upsilon_w} = 0$

where $\omega(t) = (y(t), u(t), z(t), v(t))$ is the solution corresponding to ω_0.

$\langle \mathrm{T}\omega(t), \omega(t)\rangle_{\Upsilon_w} = \int\limits_\Omega (a_1 z^2 + a_2 v^2) dx = 0$. But $a_1(x) \geq a_{1,0} > 0$ and $a_2(\mathrm{x}) \geq a_{2,0} > 0$, thus $z = 0$ and $v = 0$ in Ω.

Because $y_t = z = 0$ and $u_t = v = 0$, then y and u are constants with respect to t.

Then $y_{tt} = u_{tt} = 0$ and $Ay = Bu = 0$ (from system (5.29)).

Therefore, using Green formula

$-\int\limits_\Omega Ay\, y\, dx = \int\limits_\Omega \sum\limits_{i,j=1}^{n} a_{ij}\partial_i y \partial_j y\, dx = 0$ and $-\int\limits_\Omega Bu\, u\, dx = \int\limits_\Omega \sum\limits_{i,j=1}^{n} b_{ij}\partial_i u \partial_j u\, dx = 0.$

But,

$\int\limits_\Omega \sum\limits_{i,j=1}^{n} a_{ij}\partial_i y \partial_j y \, dx \geq a_0 \int\limits_\Omega \sum\limits_{i=1}^{n} (\partial_i y)^2 dx$ and $\int\limits_\Omega \sum\limits_{i,j=1}^{n} b_{ij}\partial_i u \partial_j u \, dx \geq b_0 \int\limits_\Omega \sum\limits_{i=1}^{n} (\partial_i u)^2 dx$,

where $a_0 > 0$ and $b_0 > 0$, therefore y and u are constants with respect to x.

Finally, $y = \chi_1$ and $u = \chi_2$, where χ_1 and χ_2 are constants.

Hence the ω-limit set contains only elements of the form $(\chi_1, \chi_2, 0, 0)$ and we find $\lim\limits_{t\to\infty}(y(t), u(t), z(t), v(t)) = (\chi_1, \chi_2, 0, 0)$.

Now, we have to find the expression of χ_1 and χ_2. We have

$y_{tt} - Ay + a_1 y_t + b u_{tt} = 0$ in $\Omega \times (0, \infty)$, $u_{tt} - Bu + a_2 u_t + b y_{tt} = 0$ in $\Omega \times (0, \infty)$

and $\partial_A y = \partial_B u = 0$ on $\Gamma \times (0, \infty)$.

This implies that $y_{tt} - Ay + a_1 y_t + b u_{tt} + u_{tt} - Bu + a_2 u_t + b y_{tt} = 0$,

which is equivalent to $(1 + b)y_{tt} + (1 + b)u_{tt} + a_1 y_t + a_2 u_t = Ay + Bu$.

Then

$$\left(\int\limits_\Omega ((1 + b)(y_t + u_t) + a_1 y + a_2 u) dx\right)' = \int\limits_\Omega Ay \, dx + \int\limits_\Omega Bu \, dx = 0.$$

Therefore, $\int\limits_\Omega ((1 + b)(y_t + u_t) + a_1 y(x, t) + a_2 u(x, t)) \, dx$ is a constant function.

Thus,

$$\int\limits_\Omega ((1 + b)y_t(x, 0) + (1 + b)u_t(x, 0) + a_1 y(x, 0) + a_2 u(x, 0)) \, dx =$$
$$\int\limits_\Omega (1 + b)(z_0 + v_0) + (a_1 y_0 + a_0 u_0) \, dx.$$

By passing to the limit where t goes to ∞ and using the fact that

$\lim\limits_{t\to\infty}(y(t), u(t), z(t), v(t)) = (\chi_1, \chi_2, 0, 0)$, we get

$$\int\limits_\Omega (0 + 0 + a_1\chi_1 + a_2\chi_2) \, dx = \int\limits_\Omega (1 + b)(z_0 + v_0) \, dx + \int\limits_\Omega (a_1 y_0 + a_0 u_0) \, dx,$$

this implies that

$$\chi_1 \left(\int\limits_\Omega a_1 \, dx\right) + \chi_2 \left(\int\limits_\Omega a_2 \, dx\right) = \left[\int\limits_\Omega (1 + b)(z_0 + v_0) \, dx + \int\limits_\Omega (a_1 y_0 + a_0 u_0) \, dx\right].$$

If $A = B$, $a_1 = a_2$ and $(y_0, z_0) = (u_0, v_0)$, then it follows from the symmetry of (5.30) that

$$\chi_1 = \chi_2 = \left(\int_\Omega a_1\,dx\right)^{-1}\left(\int_\Omega (1+b)z_0\,dx + \int_\Omega a_1 y_0\,dx\right). \qquad \blacksquare$$

Remark

(i) One can consider dynamical boundary conditions for both equations

$$\begin{cases} y_{tt}(x,t) - Ay(x,t) + a_1 y_t(x,t) + bu_{tt} = 0 & \text{in } \Omega \times (0,\infty) \\ u_{tt}(x,t) - Bu(x,t) + a_2 u_t(x,t) + by_{tt} = 0 & \text{in } \Omega \times (0,\infty) \\ m(x)y_{tt}(x,t) + \partial_A y = 0 & \text{on } \Gamma \times (0,\infty) \\ M(x)u_{tt}(x,t) + \partial_B u = 0 & \text{on } \Gamma \times (0,\infty) \\ y(x,0) = y_0(x),\ y_t(x,0) = z_0(x) & \text{in } \Omega \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x) & \text{in } \Omega \\ y_t|_\Gamma(x,0) = w_0^0(x),\ \ u_t|_\Gamma(x,0) = w_1^0(x) & \text{on } \Gamma \end{cases}$$

or static boundary condition for one equation and dynamical boundary condition for the other equation and we obtain in both cases the same results as for (5.30) with the constants χ_1 and χ_2 defined above.

(ii) We can also consider static or dynamical boundary conditions only for y, and the homogeneous Dirichlet ones for u (or the reverse). In this case, we get

$$(y(t), u(t), y_t(t), u_t(t)) \to (\chi, 0, 0, 0),$$

where

$$\chi = \left(\int_\Omega a_1\, dx\right)^{-1}\left(\int_\Omega (1+b)z_0 dx\, + \int_\Omega a_1 y_0\, dx\right).$$

(iii) Similar results can obtained for a coupled Petrovsky-Petrovsky or wave-Petrovsky systems with static or dynamical boundary conditions.

5.3 Elastic system

Let Ω be a bounded open connected set in $\mathbb{R}^n$ having a smooth boundary $\Gamma = \partial\Omega$ of class C^2. We consider the following elasticity system:

$$\begin{cases} y_{itt}(x,t) - \sum\limits_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x) y_{it}(x,t) = 0 & \text{in } \Omega \times (0,\infty),\ \forall i = 1,2,...,n \\ \sum\limits_{j=1}^{n} \sigma_{ij}(y)\nu_j = 0 & \text{on } \Gamma \times (0,\infty),\ \forall i = 1,2,...,n \\ y_i(x,0) = y_i^0(x),\ y_{it}(x,0) = z_i^0(x) & \text{in } \Omega,\ \forall i = 1,2,...,n \end{cases} \tag{5.45}$$

where $(y^0, z^0) = ((y_1^0, ..., y_n^0), (z_1^0, ..., z_n^0))$ is given initial data in $\Upsilon_e = (H^1(\Omega))^n \times (L^2(\Omega))^n$. Here $y = (y_1, ..., y_n) : \Omega \to \mathbb{R}^n$ is the solution of (5.45), $a_i(x) \in L^\infty(\Omega)$ such that there exists $a_{i,0} > 0$ for which $a_i(x) \geq a_{i,0} > 0$ $a.e.$ $x \in \Omega$, $\forall i = 1, ..., n$. Moreover, $\sigma_{ij,j}(y) = \frac{\partial \sigma_{ij}(y)}{\partial x_j}$, $\sigma_{ij}(y) = \sum\limits_{k,l=1}^{n} a_{ijkl}\varepsilon_{kl}(y)$, $\varepsilon_{ij}(y) = \frac{1}{2}(y_{i,j} + y_{j,i})$, $y_{i,j} = \frac{\partial y_i}{\partial x_j}$, $y_{j,i} = \frac{\partial y_j}{\partial x_i}$ and $a_{ijkl} \in C^1(\bar{\Omega})$ with $a_{ijkl} = a_{klij} = a_{jikl}$, $\forall i,j,k,l = 1,2,...,n$ and satisfying, for $a_0 > 0$, $\sum\limits_{i,j,k,l=1}^{n} a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq a_0 \sum\limits_{i,j=1}^{n} \varepsilon_{ij}\varepsilon_{ij}$, for all symmetric tensor ε_{ij}. For more details concerning these systems, see [15]-[18] and the references therein.

5.3.1 Preliminaries and well-posedness of the problem

In this subsection, we study the existence and uniqueness of the solutions of the system (5.45). Let us consider the state space

$$\Upsilon_e = \left(H^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n,$$

equipped with the inner product, for any $y = (y_1, ..., y_n)$, $z = (z_1, ..., z_n)$,

$\tilde{y} = (\tilde{y}_1, ..., \tilde{y}_n)$ and $\tilde{z} = (\tilde{z}_1, ..., \tilde{z}_n)$,

$$\begin{aligned} \langle (y,z), (\tilde{y}, \tilde{z}) \rangle_{\Upsilon_e} = &\int_\Omega \left(\sum_{i,j=1}^{n} \sigma_{ij}(y) \varepsilon_{ij}(\tilde{y}) \right) dx + \int_\Omega (\sum_{i=1}^{n} z_i \tilde{z}_i) dx \\ &+ \varepsilon \sum_{i=1}^{n} \left[\left(\int_\Omega (z_i + a_i y_i)\, dx \right) \left(\int_\Omega (\tilde{z}_i + a_i \tilde{y}_i)\, dx \right) \right], \end{aligned} \tag{5.46}$$

where $\varepsilon > 0$ is a constant to be determined. The first result is stated in the following proposition.

Proposition 5.4 *The state space $\Upsilon_e = \left(H^1(\Omega)\right)^n \times \left(L^2(\Omega)\right)^n$, endowed with the inner product (5.46) is a Hilbert space provided that ε is small enough.*

Proof. It is sufficient to show that the norm $\|\,.\,\|_{\Upsilon_e}$ induced by the inner product (5.46) is equivalent to the usual one $\|\,.\,\|_{(H^1(\Omega))^n \times (L^2(\Omega))^n}$; that is, we prove the existence of two positive constants K and $\tilde{K}$ such that, for any

$y = (y_1, ..., y_n) \in \Upsilon_e$ and $z = (z_1, ..., z_n) \in \Upsilon_e$,

$$K \left\|(y,z)\right\|^2_{(H^1(\Omega))^n \times (L^2(\Omega))^n} \leq \left\|(y,z)\right\|^2_{\Upsilon_e} \leq \tilde{K} \left\|(y,z)\right\|^2_{(H^1(\Omega))^n \times (L^2(\Omega))^n}. \tag{5.47}$$

On one hand,

$$\|(y,z)\|^2_{\Upsilon_e} = \int_\Omega \left(\sum_{i,j=1}^{n} \sigma_{ij}(y)\varepsilon_{ij}(y)\right) dx \,+\, \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx \,+\, \varepsilon \sum_{i=1}^{n} \left(\int_\Omega (z_i + a_i y_i)\, dx\right)^2$$
$$= \int_\Omega \left(\sum_{i,j=1}^{n} \sum_{k,l=1}^{n} a_{ijkl}\varepsilon_{kl}(y)\varepsilon_{ij}(y)\right) dx \,+\, \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx \,+\, \varepsilon \sum_{i=1}^{n} \left(\int_\Omega (z_i + a_i y_i)\, dx\right)^2.$$

Applying Hölder′s and Young's inequalities, we get

$$\|(y,z)\|^2_{\Upsilon_e} \le \tfrac{1}{2} \int_\Omega \left[\sum_{i,j,k,l=1}^{n} \sup_{x\in\Omega} |a_{ijkl}(x)| \left(\varepsilon^2_{ij}(y) + \varepsilon^2_{kl}(y)\right)\right] dx + \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx$$
$$+\varepsilon\, vol(\Omega) \sum_{i=1}^{n} \int_\Omega (z_i + a_i y_i)^2 dx.$$

Let $\tilde{a} = \max\limits_{i,j,k,l} \sup\limits_{x\in\Omega} |a_{ijkl}(x)|$ and using the fact that

$2\,|a_i|\,|y_i|\,|z_i| \le \|a_i\|_\infty (y_i^2 + z_i^2)$, we get

$$\|(y,z)\|^2_{\Upsilon_e} \le \tfrac{\tilde{a}}{2} \int_\Omega \left[\sum_{i,j,k,l=1}^{n} \left(\varepsilon^2_{ij}(y) + \varepsilon^2_{kl}(y)\right)\right] dx \,+ \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx$$
$$+\varepsilon\, vol(\Omega) \sum_{i=1}^{n} \int_\Omega \left(z_i^2 + \|a_i\|^2_\infty\, y_i^2 + \|a_i\|_\infty y_i^2 + \|a_i\|_\infty z_i^2\right) dx$$
$$= \tfrac{\tilde{a}}{2} \int_\Omega \left[n^2 \sum_{i,j=1}^{n} \varepsilon^2_{ij}(y) + n^2 \sum_{k,l=1}^{n} \varepsilon^2_{kl}(y)\right] dx + \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx$$
$$+\varepsilon\, vol(\Omega) \sum_{i=1}^{n} \int_\Omega \left((1 + \|a_i\|_\infty)\, z_i^2 + \|a_i\|_\infty \left(\|a_i\|_\infty + 1\right) y_i^2\right) dx.$$

Then,

$$\|(y,z)\|^2_{\Upsilon_e} \le \tfrac{\tilde{a}}{2} \int_\Omega \left[\tfrac{n^2}{4} \sum_{i,j=1}^{n} (y_{i,j} + y_{j,i})^2 + \tfrac{n^2}{4} \sum_{k,l=1}^{n} (y_{k,l} + y_{l,k})^2\right] dx$$
$$+ \sum_{i=1}^{n} \left[1 + \varepsilon\, vol(\Omega) \left(\|a_i\|_\infty + 1\right)\right] \int_\Omega z_i^2 dx \,+ \sum_{i=1}^{n} \left[\varepsilon\, vol(\Omega) \|a_i\|_\infty \left(\|a_i\|_\infty + 1\right)\right] \int_\Omega y_i^2 dx$$
$$= \tfrac{\tilde{a}n^2}{8} \int_\Omega \left[\sum_{i,j=1}^{n} \left(\tfrac{\partial y_i}{\partial x_j} + \tfrac{\partial y_j}{\partial x_i}\right)^2 + \sum_{k,l=1}^{n} \left(\tfrac{\partial y_k}{\partial x_l} + \tfrac{\partial y_l}{\partial x_k}\right)^2\right] dx$$
$$+ \sum_{i=1}^{n} \left[1 + \varepsilon\, vol(\Omega) \left(\|a_i\|_\infty + 1\right)\right] \int_\Omega z_i^2 dx + \sum_{i=1}^{n} \left[\varepsilon\, vol(\Omega) \|a_i\|_\infty \left(\|a_i\|_\infty + 1\right)\right] \int_\Omega y_i^2 dx.$$

Therefore,

$$\|(y,z)\|^2_{\Upsilon_e} \leq \frac{\tilde{a}n^2}{8}\int_\Omega \sum_{i,j=1}^n \left[2\left(\frac{\partial y_i}{\partial x_j}\right)^2 + 2\left(\frac{\partial y_j}{\partial x_i}\right)^2\right] dx + \frac{\tilde{a}n^2}{8}\int_\Omega \sum_{k,l=1}^n \left[2\left(\frac{\partial y_k}{\partial x_l}\right)^2 + 2\left(\frac{\partial y_l}{\partial x_k}\right)^2\right] dx$$
$$+ \sum_{i=1}^n \left[1+\varepsilon\, vol(\Omega)\left(\|a_i\|_\infty + 1\right)\right]\int_\Omega z_i^2 dx + \sum_{i=1}^n \left[\varepsilon\, vol(\Omega)\|a_i\|_\infty\left(\|a_i\|_\infty + 1\right)\right]\int_\Omega y_i^2 dx.$$

$$\|(y,z)\|^2_{\Upsilon_e} \leq \frac{\tilde{a}n^2}{4}\int_\Omega \left[\sum_{i=1}^n |\nabla y_i|^2 + \sum_{j=1}^n |\nabla y_j|^2\right] dx + \frac{\tilde{a}n^2}{4}\int_\Omega \left[\sum_{k=1}^n |\nabla y_k|^2 + \sum_{l=1}^n |\nabla y_l|^2\right] dx$$
$$+ \sum_{i=1}^n \left[1+\varepsilon\, vol(\Omega)\left(\|a_i\|_\infty + 1\right)\right]\int_\Omega z_i^2 dx + \sum_{i=1}^n \left[\varepsilon\, vol(\Omega)\|a_i\|_\infty\left(\|a_i\|_\infty + 1\right)\right]\int_\Omega y_i^2 dx.$$

Then,

$$\|(y,z)\|^2_{\Upsilon_e} \leq \tilde{a}n^2 \int_\Omega \sum_{i=1}^n |\nabla y_i|^2 dx + \sum_{i=1}^n \left[1+\varepsilon\, vol(\Omega)\left(\|a_i\|_\infty + 1\right)\right]\int_\Omega z_i^2 dx$$
$$+ \sum_{i=1}^n \left[\varepsilon\, vol(\Omega)\|a_i\|_\infty\left(\|a_i\|_\infty + 1\right)\right]\int_\Omega y_i^2 dx.$$

Let $\alpha_{i,0} = 1+\varepsilon\, vol(\Omega)\left(1+\|a_i\|_\infty\right)$, $\beta_{i,0} = \varepsilon\, vol(\Omega)\,\|a_i\|_\infty\left(\|a_i\|_\infty + 1\right)$ and

$\tilde{K} = \max\{\tilde{a}n^2, \alpha_{i,0}, \beta_{i,0}\}$. Consequently,

$$\|(y,z)\|^2_{\Upsilon_e} \leq \tilde{K}\,\|(y,z)\|^2_{(H^1(\Omega))^n\times(L^2(\Omega))^n}. \tag{5.48}$$

On the other hand,

$$\|(y,z)\|^2_{\Upsilon_e} \geq a_0\int_\Omega \left(\sum_{i,j=1}^n \varepsilon_{ij}(y)\varepsilon_{ij}(y)\right) dx + \int_\Omega \left(\sum_{i=1}^n z_i^2\right) dx$$
$$+\varepsilon\sum_{i=1}^n \left[\left(\int_\Omega z_i dx\right)^2 + \left(\int_\Omega a_i y_i\, dx\right)^2 + 2\left(\int_\Omega z_i dx\right)\left(\int_\Omega a_i y_i\, dx\right)\right].$$

But, $2\varepsilon\left(\int_\Omega z_i dx\right)\left(\int_\Omega a_i y_i\, dx\right) \geq -\varepsilon\left[\delta\left(\int_\Omega a_i y_i\, dx\right)^2 + \frac{1}{\delta}\left(\int_\Omega z_i dx\right)^2\right],\ \ \forall\delta > 0.$

Then,

$$\|(y,z)\|^2_{\Upsilon_e} \geq a_0 \int\limits_\Omega \left(\sum_{i,j=1}^n \varepsilon_{ij}(y)\varepsilon_{ij}(y) \right) dx + \int\limits_\Omega \left(\sum_{i=1}^n z_i^2 \right) dx$$
$$+ \sum_{i=1}^n \left[\varepsilon\,(1-\delta) \left(\int\limits_\Omega a_i y_i \, dx \right)^2 + \varepsilon \left(1 - \tfrac{1}{\delta}\right) \left(\int\limits_\Omega z_i dx \right)^2 \right].$$

Using generalized Poincaré's inequality [3], we can prove that there exists a positive constant c_0 such that

$$\int\limits_\Omega \left(\sum_{i=1}^n y_i^2 \right) dx \ \leq c_0 \left[\int\limits_\Omega \left(\sum_{i,j=1}^n \sigma_{ij}(y)\varepsilon_{ij}(y) \right) \ dx + \sum_{i=1}^n \left(\int\limits_\Omega a_i y_i dx \right)^2 \right], \ \forall y \in \left(H^1(\Omega)\right)^n, \tag{5.49}$$

which implies that

$$\sum_{i=1}^n \left(\int\limits_\Omega a_i y_i dx \right)^2 \geq \tfrac{1}{c_0} \int\limits_\Omega \left(\sum_{i=1}^n y_i^2 \right) dx \ - \int\limits_\Omega \left(\sum_{i,j=1}^n \sigma_{ij}(y)\varepsilon_{ij}(y) \right) \ dx.$$

The proof of (5.49) is based on the Korn's inequality: there exists $\tilde{c}_0 > 0$ such that

$$\int\limits_\Omega \left(\sum_{i=1}^n |\partial_i y_j|^2 + \sum_{i=1}^n y_i^2 \right) dx \leq \tilde{c}_0 \left[\int\limits_\Omega \left(\sum_{i,j=1}^n \varepsilon_{ij}(y)\varepsilon_{ij}(y) \right) dx + \int\limits_\Omega \left(\sum_{i=1}^n a_i y_i^2 \right) dx \right], \quad \forall y \in \left(H^1(\Omega)\right)^n. \tag{5.50}$$

Hence, for $0 < \delta < 1$ (so $1 - \frac{1}{\delta} > 0$ and $1 - \delta > 0$)

$$\|(y,z)\|^2_{\Upsilon_e} \geq a_0 \sum_{i,j=1}^n \int\limits_\Omega \varepsilon_{ij}^2(y) \, dx + \int\limits_\Omega \left(\sum_{i=1}^n z_i^2 \right) dx + \tfrac{\varepsilon(1-\delta)}{c_0} \int\limits_\Omega \left(\sum_{i=1}^n y_i^2 \right) dx$$
$$-\varepsilon(1-\delta) \int\limits_\Omega \left(\sum_{i,j=1}^n \sigma_{ij}(y)\varepsilon_{ij}(y) \right) \ dx + \left(1 - \varepsilon\tfrac{1}{\delta}\right) vol(\Omega) \int\limits_\Omega \left(\sum_{i=1}^n z_i^2 \right) dx.$$

Let $\tilde{a} = \max\limits_{i,j,k,l} \sup\limits_{x\in\Omega} |a_{ijkl}(x)|$ and applying Young's inequality, we get

$$\|(y,z)\|^2_{\Upsilon_e} \geq a_0 \sum_{i,j=1}^n \int\limits_\Omega \varepsilon_{ij}^2(y) \, dx + \left(1 + \varepsilon \left(1 - \tfrac{1}{\delta}\right) vol(\Omega)\right) \int\limits_\Omega \left(\sum_{i=1}^n z_i^2 \right) dx$$
$$+ \tfrac{\varepsilon(1-\delta)}{c_0} \int\limits_\Omega \left(\sum_{i=1}^n y_i^2 \right) dx - \tilde{a}\varepsilon(1-\delta) \int\limits_\Omega \left(\sum_{i,j,k,l=1}^n \left(\varepsilon_{ij}^2(y) + \varepsilon_{kl}^2(y) \right) \right) \ dx.$$

Thus,

$$\|(y,z)\|^2_{\Upsilon_e} \geq a_0 \sum_{i,j=1}^{n} \int_\Omega \varepsilon_{ij}^2(y)\,dx + \left(1+\varepsilon\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\right) \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx$$
$$+\tfrac{\varepsilon(1-\delta)}{c_0} \int_\Omega \left(\sum_{i=1}^{n} y_i^2\right) dx - 2\tilde{a}\varepsilon(1-\delta)n^2 \int_\Omega \left(\sum_{i,j=1}^{n} \varepsilon_{ij}^2(y)\right) dx.$$

Then,

$$\|(y,z)\|^2_{\Upsilon_e} \geq (a_0 - 2\tilde{a}\varepsilon(1-\delta)n^2) \sum_{i,j=1}^{n} \int_\Omega \varepsilon_{ij}^2(y)\,dx + \left(1+\varepsilon\left(1-\tfrac{1}{\delta}\right) vol(\Omega)\right) \int_\Omega \left(\sum_{i=1}^{n} z_i^2\right) dx$$
$$+\tfrac{\varepsilon(1-\delta)}{c_0} \int_\Omega \left(\sum_{i=1}^{n} y_i^2\right) dx.$$

Let $\alpha_1 = a_0 - 2\tilde{a}\varepsilon(1-\delta)n^2$ and $\alpha_2 = 1 - \left(\frac{1}{\delta}\right) vol(\Omega)$, then we get

$$\|(y,z)\|^2_{\Upsilon_e} \geq \min\left\{\alpha_1, \alpha_2, \tfrac{\varepsilon(1-\delta)}{c_0}\right\} \int_\Omega \left(\sum_{i,j=1}^{n} \varepsilon_{ij}^2(y) + \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} z_i^2\right) dx.$$

Therefore,

$\|(y,z)\|^2_{\Upsilon_e} \geq \alpha_3 \left[\frac{1}{\tilde{c}_0} \|y\|^2_{(H^1(\Omega))^n} + \|z\|^2_{(L^2(\Omega))^n}\right]$, where $\alpha_3 = \min\left\{\alpha_1, \alpha_2, \frac{\varepsilon(1-\delta)}{c_0}\right\}$.

Finally,

$$\|(y,z)\|^2_{\Upsilon_e} \geq K \|(y,z)\|^2_{(H^1(\Omega))^n \times (L^2(\Omega))^n}, \tag{5.51}$$

where $K = \alpha_3 \min\left\{\frac{1}{\tilde{c}_0}, 1\right\}$.

From (5.48) and (5.51), we get that

$$K \|(y,z)\|^2_{(H^1(\Omega))^n \times (L^2(\Omega))^n} \leq \|(y,z)\|^2_{\Upsilon_e} \leq \tilde{K} \|(y,z)\|^2_{(H^1(\Omega))^n \times (L^2(\Omega))^n}.$$

Therefore, the state space $\|(y,z)\|_{\Upsilon_e}$ endowed with the inner product (5.46) is a Hilbert space. ∎

We turn now to the formulation of the system (5.45) in an abstract form in Υ_e. Let $z(t) = y_t(t)$ and $\Phi(t) = (y(t), z(t))$.

Then, the system (5.45) can be written as

$$\begin{cases} \Phi_t(t) + \mathrm{T}_e\Phi(t) = 0, \\ \\ \Phi(0) = \Phi_0 = (y^0, z^0), \end{cases} \tag{5.52}$$

where T_e is an unbounded linear operator defined by:

$$\mathrm{T}_e\,(y,z) = \left(-z, \left(-\sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x) z_i\right)_i\right), \qquad \forall (y,z) \in D(\mathrm{T}_e) \tag{5.53}$$

and

$$\begin{aligned}
D(\mathrm{T}_e) &= \left\{ (y,z) \in \Upsilon_e \;:\; \mathrm{T}_e(y,z) \in \Upsilon_e \text{ and} \sum_{j=1}^{n} \sigma_{ij}(y)\nu_j = 0 \text{ on } \Gamma,\ i = 1,...,n \right\} \\
&= \left\{ \begin{array}{l} (y,z) \in (H^1(\Omega))^n \times (L^2(\Omega))^n : \\ \left(-z, \left(-\sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x) z_i\right)_i\right) \in (H^1(\Omega))^n \times (L^2(\Omega))^n, \\ \text{and} \sum_{j=1}^{n} \sigma_{ij}(y)\nu_j = 0 \text{ on } \Gamma,\ i = 1,...,n \end{array} \right\} \\
&= \left\{ \begin{array}{l} (y,z) \in (H^1(\Omega))^n \times (L^2(\Omega))^n :\ -z \in (H^1(\Omega))^n, \\ \left(-\sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x) z_i\right)_i \in (L^2(\Omega))^n, \\ \text{and} \sum_{j=1}^{n} \sigma_{ij}(y)\nu_j = 0 \text{ on } \Gamma,\ i = 1,...,n \end{array} \right\} \\
&= \left\{ (y,z) \in \left(H^2(\Omega)\right)^n \times \left(H^1(\Omega)\right)^n ;\ \text{and} \sum_{j=1}^{n} \sigma_{ij}(y)\nu_j = 0 \text{ on } \Gamma,\ i = 1,...,n \right\}.
\end{aligned} \tag{5.54}$$

We prove that T_e is maximal monotone operator.

We have, for any $(y,z) \in D(\mathrm{T}_e)$

$$\langle \mathrm{T}_e(y,z),(y,z)\rangle_{\Upsilon_e} = \left\langle \left(-z, -\sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a(x)z\right)_i, (y,z) \right\rangle_{\Upsilon_e}$$

$$= -\int_\Omega \left(\sum_{i,j=1}^{n} \sigma_{ij}(z)\varepsilon_{ij}(y)\right) dx \; + \sum_{i=1}^{n}\int_\Omega \left(- \sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x)z_i\right) z_i \; dx$$

$$+\varepsilon \sum_{i=1}^{n} \left[\left(\int_\Omega (- \sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x)z_i - a_i(x)z_i)\, dx \right) \; \left(\int_\Omega (z_i + a_i(x)y_i)\, dx \right) \right].$$

By applying the Green's formula and the fact that $\sum_{j=1}^{n} \sigma_{ij}\nu_j = 0$ on Γ,

$\forall i = 1, 2, ..., n$, we find

$$\sum_{i=1}^{n}\int_\Omega \left(- \sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x)z_i - a_i(x)z_i \right) dx = - \sum_{i=1}^{n}\int_\Omega \left(\sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) \right) dx$$

$$= - \sum_{i=1}^{n}\int_\Gamma \left(\sum_{j=1}^{n} \sigma_{ij}(y)\nu_j \right) d\sigma = 0,$$

$$-\int_\Omega \left(\sum_{i,j=1}^{n} \sigma_{ij}(z)\varepsilon_{ij}(y)\right) dx = -\int_\Omega \left(\sum_{i,j=1}^{n}\sum_{k,l=1}^{n} a_{ijkl}\varepsilon_{kl}(z)\varepsilon_{ij}(y)\right) dx$$

$$= -\tfrac{1}{4} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{ijkl}(\partial_l z_k + \partial_k z_l)(\partial_j y_i + \partial_i y_j)dx$$

$$= -\tfrac{1}{4} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{ijkl}\, \partial_l z_k(\partial_j y_i + \partial_i y_j)dx - \tfrac{1}{4} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{ijkl}\, \partial_k z_l(\partial_j y_i + \partial_i y_j)dx$$

$$= -\tfrac{1}{4} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{klij}\, \partial_j z_i(\partial_k y_l + \partial_l y_k)dx - \tfrac{1}{4} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{klij}\, \partial_j z_i(\partial_k y_l + \partial_l y_k)dx$$

$$= -\tfrac{1}{2} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{klij}\, \partial_j z_i(\partial_k y_l + \partial_l y_k)dx = -\tfrac{1}{2} \sum_{i,j,k,l=1}^{n} \int_\Omega a_{ijkl}\, \partial_j z_i(\partial_k y_l + \partial_l y_k)dx,$$

and

$$-\int_\Omega \left(\sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t)z_i\right) dx = -\int_\Gamma \left(\sum_{j=1}^{n} \sigma_{ij}(y)\nu_i z_i\right) d\sigma + \int_\Omega \left(\sum_{i,j=1}^{n} \sigma_{ij}(y)\partial_j z_i\right) dx$$

$$= \tfrac{1}{2} \int_\Omega \sum_{i,j,k,l=1}^{n} a_{ijkl}\, \partial_j z_i(\partial_k y_l + \partial_l y_k)dx.$$

Then, we get $\langle \mathrm{T}_e(y,z),(y,z)\rangle_{\Upsilon_e} = \sum_{i=1}^{n}\int_\Omega a_i(x)z_i^2 dx \geq 0$, so we conclude that T_e is monotone.

Now, we want to prove that $Id + \mathrm{T}_e$ is surjective.

Let $(f_1, f_2) = ((f_{11}, ..., f_{1n}), (f_{21}, ..., f_{2n})) \in \Upsilon_e$. We want to find $(y,z) \in D(\mathrm{T}_e)$ such that $(Id + \mathrm{T}_e)(y,z) = (f_1, f_2)$.

We have $(y,z)+\mathrm{T}_e(y,z)=(f_1,f_2)$. This means that

$$(y,z)+\left(-z_i,\left(-\sum_{j=1}^{n}\sigma_{ij,j}(y)(x,t)+a_i(x)z_i\right)_i\right)=(f_{i1},f_{i2}). \tag{5.55}$$

The first equation of (5.55) implies that $y-z=f_1$; that is $y_i-z_i=f_{1i}$, so $z_i=y_i-f_{1i}$, $i=1,...,n$.

The second equation of (5.55) becomes $-\sum\limits_{j=1}^{n}\sigma_{ij,j}(y)(x,t)+a_i(x)z_i+z_i=f_{2i}$; that is, $-\sum\limits_{j=1}^{n}\sigma_{ij,j}(y)(x,t)+(a_i(x)+1)(y_i-f_{1i})=f_{2i}$, which is equivalent to

$$-\sum_{j=1}^{n}\sigma_{ij,j}(y)(x,t)+(a_i+1)y_i=f_i, \qquad \text{where } f_i=(a_i+1)f_{1i}+f_{2i}\in L^2(\Omega). \tag{5.56}$$

As we did previously in the case of the wave equation with static boundary conditions (Subsection 2.2) and dynamic boundary conditions (Subsection 3.2), using variation formulation and Lax-Milgram theorem [4], we deduce that (5.56) has a unique solution $y\in(H^2(\Omega))^n$ and satisfying $\sum\limits_{j=1}^{n}\sigma_{ij}(y)\nu_j=0$ on Γ, $\forall i=1,2,...,n$, therefore $z=y-f_1$ exists in $(H^1(\Omega))^n$ and (5.56) holds.

We conclude that (5.55) has a unique solution $(y,z)\in D(\mathrm{T}_e)$; that is $Id+\mathrm{T}_e$ is surjective. Finally, we conclude that T_e is maximal monotone operator.

By Hille-Yosida theorem (see [4], [27], [34] and [36]), we get the following:

1) For all $\Phi_0\in\Upsilon_e$, there exists a unique $\Phi\in C(\mathbb{R}^+,\Upsilon_e)$ solution of (5.52). This implies that there exists a unique y is a solution of (5.45) satisfying

$y \in C\left(\mathbb{R}^+, \left(H^1(\Omega)\right)^n\right)$, $y_t \in C\left(\mathbb{R}^+, \left(L^2(\Omega)\right)^n\right) \Leftrightarrow y \in C^1\left(\mathbb{R}^+, \left(L^2(\Omega)\right)^n\right) \cap C\left(\mathbb{R}^+, \left(H^1(\Omega)\right)^n\right)$.

2) If $\Phi_0 \in D(\mathrm{T}_e)$, then $\Phi \in C^1(\mathbb{R}^+, \Upsilon_e) \cap C(\mathbb{R}^+, D(\mathrm{T}_e))$; that is, for all $(y^0, z^0) \in D(\mathrm{T}_e)$, $(y, z) \in C^1\left(\mathbb{R}^+, (H^1(\Omega))^n \times (L^2(\Omega))^n\right) \cap C\left(\mathbb{R}^+, (H^2(\Omega))^n \times (H^1(\Omega))^n\right)$, solution of (5.45), thus

$y \in C^1\left(\mathbb{R}^+, (H^1(\Omega))^n\right) \cap C\left(\mathbb{R}^+, (H^2(\Omega))^n\right)$, $z \in C^1\left(\mathbb{R}^+, (L^2(\Omega))^n\right) \cap C\left(\mathbb{R}^+, (H^1(\Omega))^n\right)$

$\Leftrightarrow y \in C^2\left(\mathbb{R}^+, (L^2(\Omega))^n\right) \cap C^1\left(\mathbb{R}^+, (H^1(\Omega))^n\right) \cap C\left(\mathbb{R}^+, (H^2(\Omega))^n\right)$, since $z = y_t$.

5.3.2 Stabilization of the problem

In this subsection, we prove the following stability result.

Definition 5.4 *The ω-limit set is*

$$\omega(y^0, z^0) = \left\{ \begin{array}{l} (\omega_1, \omega_2) \in \Upsilon_e \,:\, \exists \{t_n\} \text{ an increasing sequence of positive numbers;} \\ \\ \lim\limits_{n\to\infty} \|(y(t_n), z(t_n)) - (\omega_1, \omega_2)\|_{\Upsilon_e} = 0 \end{array} \right\}.$$

Theorem 5.4 *For any initial data $\Phi_0 = (y^0, z^0) \in \Upsilon_e$, the solution $\Phi(t) = (y(t), z(t)) \to (\chi, 0)$ in Υ_e as $t \to \infty$ where*

$\chi = (\chi_1, \chi_2, ..., \chi_n)$ and $\sum\limits_{i=1}^{n} \chi_i \left(\int\limits_\Omega a_i(x)\, dx \right) = \sum\limits_{i=1}^{n} \int\limits_\Omega (z_i^0 + a_{i,0} y_i^0) dx$;

that is

$$\lim_{t\to\infty} \|(y(t), z(t)) - (\chi, 0)\|^2_{\Upsilon_e} = 0.$$

Proof. Applying LaSalle's principle [24], we have:

i) $\omega(y^0, z^0) \neq \phi$, $\forall (y_i^0, z_i^0) \in \Upsilon_e$ and it is a compact set.

ii) $\omega(y^0, z^0)$ is invariant under the semi-group $S(t)$.

iii) Let $(y(t), z(t)) = S(t)(y^0, z^0)$ be a solution of (5.52), then

$\lim_{t\to\infty}(y_i(t), z_i(t)) \in \omega(y_i^0, z_i^0)$.

iv) $\omega(y^0, z^0) \subset D(\mathrm{T}_e)$.

v) $t \to \|S(t)\omega\|^2_{\Upsilon_e}$ is a constant function for any $(\omega_1, \omega_2) \in \omega(y^0, z^0)$.

We want to prove that $(y(t), z(t)) \to (\chi, 0)$, as $t \to \infty$, where $\chi = (\chi_1, ..., \chi_n)$.

From (iii), it is sufficient to prove that $\omega(y^0, z^0)$ contains only elements of the form $(\chi, 0)$.

Let $\omega_0 \in \omega(y^0, z^0)$, we prove that $\omega_0 = (\chi, 0)$. We have

$\frac{d}{dt}\left(\|S(t)\omega_0\|^2_{\Upsilon_e}\right) = 0 \Rightarrow \left\langle \frac{d}{dt}\left(S(t)\omega_0\right), S(t)\omega_0\right\rangle_{\Upsilon_e} = 0 \Rightarrow \left\langle \frac{d}{dt}\omega(t), \omega(t)\right\rangle_{\Upsilon_e} = 0,$

where $\omega(t) = (y(t), z(t))$ is the solution of (5.52) corresponding to ω_0

$\langle \mathrm{T}\omega(t), \omega(t)\rangle_{\Upsilon_e} = \int\limits_\Omega \sum\limits_{i=1}^n a_i z_i^2\,dx = 0$. But $a_i(x) \geq a_{i,0} > 0$, thus $z_i = 0$ on Ω, $\forall i = 1, ..., n$.

Because $z_i = y_{it} = 0$, then y_i is a constant with respect to t, $\forall\, i = 1, 2, ...n$; that is, y is a constant with respect to t.

In addition, $y_{itt} = 0$ and $\sum\limits_{j=1}^n \sigma_{ij,j}(y)(x,t) = 0$ (from system (5.45)).

Therefore, using Green's formula

$$-\int\limits_\Omega \sum_{j=1}^n \sigma_{ij,j}(y)(x,t)\varepsilon_{ij}(y)\,dx = -\int\limits_\Gamma \sum_{j=1}^n \sigma_{ij}(y)\nu_j\varepsilon_{ij}(y)\,d\sigma + \int\limits_\Omega \sum_{i,j=1}^n \sigma_{ij}(y)\varepsilon_{ij}(y)dx$$
$$= \int\limits_\Omega \sum_{i,j,k,l=1}^n a_{ijkl}\varepsilon_{k,l}(y)\varepsilon_{ij}(y)\,dx = 0,$$

by Korn's inequality (5.50) we get that y_i is a constant with respect to x, $\forall\, i = 1, 2, ..., n$; that is, $y = (y_1, ..., y_n)$ is a constant with respect to x.

Finally, $y = \chi$, where $\chi = (\chi_1, ..., \chi_n)$ is a constant.

Hence the ω-limit set contains only elements of the form $(\chi, 0)$, where χ is a constant, and we find that $\lim_{t\to\infty}(y(t),z(t)) = (\chi, 0)$.

Now, we have to find the expression of χ. Because

$y_{itt}(x,t) - \sum_{j=1}^{n} \sigma_{ij,j}(y)(x,t) + a_i(x) y_{it}(x,t) = 0$ in $\Omega\times(0,\infty)$ and $\sum_{j=1}^{n}\sigma_{ij}(y)\nu_j = 0$ on $\Gamma\times(0,\infty)$, then $\left(\int_\Omega \sum_{i=1}^{n}(y_{it}+a_i y_i)dx\right)' = \int_\Omega \sum_{j=1}^{n}\sigma_{ij,j}(y)(x,t)\,dx = \int_\Gamma \sum_{j=1}^{n}\sigma_{ij}(y)\nu_j d\sigma = 0\,,$ therefore $\int_\Omega \sum_{i=1}^{n}(y_{it}+a_i y_i)dx$ is a constant function. Thus,

$$\begin{aligned}\int_\Omega \sum_{i=1}^{n}\left(y_{it}(x,t)+a_i(x)y_i(x,t)\right)\,dx &= \int_\Omega \sum_{i=1}^{n}\left(y_{it}(x,0)+a_i(x)y_i(x,0)\right)\,dx\\ &= \int_\Omega \sum_{i=1}^{n}\left(z_i^0 + a_i y_i^0\right)\,dx.\end{aligned}$$

By passing to the limit where t goes to ∞, and using the fact that

$\lim_{t\to\infty}(y(t),z(t)) = (\chi,0)$, we get

$\int_\Omega \sum_{i=1}^{n}(0 + a_i\,\chi_i)\,dx = \int_\Omega \sum_{i=1}^{n}(z_i^0 + a_i\,y_i^0)\,dx$, this implies that

$\sum_{i=1}^{n}\chi_i\left(\int_\Omega a_i\,dx\right) = \int_\Omega \sum_{i=1}^{n}(z_i^0 + a_i\,y_i^0)\,dx.$

If $a_i = a_j$, $y_i^0 = y_j^0$ and $z_i^0 = z_j^0$, $\forall i,j = 1,2,...,n$, then by symmetry of (5.45) we have

$$\chi_i = \left(\int_\Omega a_i\,dx\right)^{-1}\int_\Omega (z_i^0 + a_i(x)\,y_i^0)\,dx. \qquad \blacksquare$$

Remark

We can consider static conditions for y_i, $i = 1,2,...,r$, dynamical boundary conditions for y_i, $i = r+1,...,p$, and the homogeneous Dirichlet ones for y_i, $i = p+1,...,n$, where $0 \le r \le p \le n$; that is

$$\begin{cases} \sum_{j=1}^{n} \sigma_{ij}(y)\nu_j = 0, & \text{on}\,\Gamma \times (0,\infty),\ \forall i = 1,...,r, \\ \sum_{j=1}^{n} \sigma_{ij}(y)\nu_j + m_i y_{itt} = 0, & \text{on}\,\Gamma \times (0,\infty),\ \forall i = r+1,...,p, \\ y_i = 0 & \text{on}\,\Gamma \times (0,\infty),\ \forall i = p+1,...,n. \end{cases}$$

In this case, $(y(t), y_t(t)) \to (\chi, 0)$ as $t \to \infty$, where

$\chi = (\chi_1, \chi_2, ..., \chi_n)$.

$\chi_i = 0$ for $i = p+1, ..., n$, and

$$\sum_{i=1}^{p} \chi_i \left(\int_\Omega a_i(x)\ dx \right) = \int_\Omega \sum_{i=1}^{p} (z_i^0 + a_i(x) y_i^0)\ dx.$$

CHAPTER 6

OPEN PROBLEMS

1. We have proved that $(y, y_t) \to (\chi, 0)$ in Υ as $t \to \infty$. However, we do not know how is the convergence (exponential ? Polynomial ?...) ?

2. In [13] the nonlinear damping control was considered only in the case of dynamical Neumann boundary conditions. What about the case of static Neumann boundary conditions ?

3. What about the case of unbounded domain (for example $\mathbb{R}^n$ or exterior domains) ?

4. In the case of coupled systems. What about if we consider different partitions (Γ_0, Γ_1) of Γ for both equations ?

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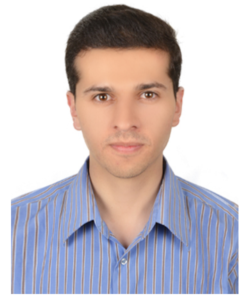
Vitae



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